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บทคัดย่อ

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Academic Article

Analysis of the rate of return for saving for retirement using present values

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Abstract

This article demonstrates the applications of mathematics to the decision making of investing in savings with financial institutions that provides good returns for retirement saving programs. Mathematical knowledge about interests and current values can be used to analyze and compare the rate of return for each retirement saving project of a financial institution. To guide and help you decide on a good saving program, there are demonstrations of real analysis methods with financial institution' retirement saving programs to compare the best returns under the same conditions. The results show that mathematics enables us to have another way for more effective financial planning.

Keywords: Saving for retirement, Present value, Interest rate

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Introduction

When society is entering an aging society without income from work, management and planning for retirement savings are important to ensure the adequacy of retirement income to be able to live after retirement with quality and stability. In general, retirees will not have income from regular work and not have any other income that makes it stable. Therefore, saving and investing in working age are important factors in the adequacy and sustainability of lives after retirement (Lusardi, 2009; Rappaport & Ward, 2017).

The World Bank, therefore, has established additional compulsory and voluntary saving schemes and the provision of a welfare system to support the retirement lives under the five frameworks called “The Five Pillar Framework” (Holzmann, Hinz, & Dorfman, 2008) and shown as follows:

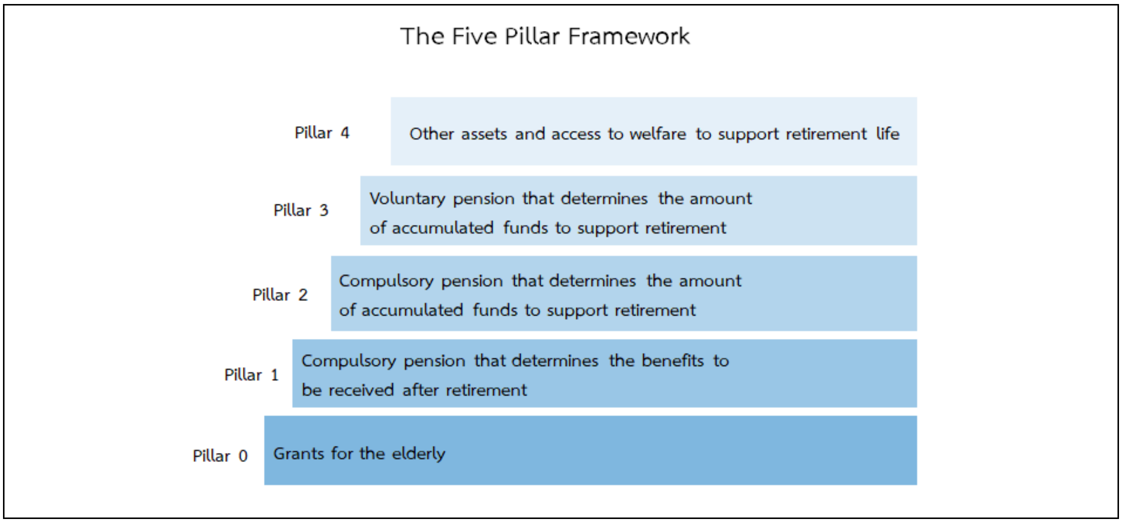

Figure 1. The five pillar framework on the pension system for retirement.

Based on the five pillar framework, all financial institutions offer financial products which are the income generating plan of the financial products after retirement such as insurance for accumulating assets, the retirement mutual funds, saving for retirement projects, and so on which must be well planned regarding the purchases of such products (Clark, D'Ambrosio, McDermed, & Sawant, 2006; Dong, Halen, Moore, & Zeng, 2019).

Mathematics has a role to help making decisions in order to choose financial products more easily and efficiently, especially in terms of saving for retirements. The case of retirement at the age of 60 years is expected to live after retirement for another 20 years. Retirees want to have retirement costs of 21000 baht and start saving money at the age of 40 years. How much is the monthly saving plan required? Mathematics can help to answer this question. There are researches on the applications of mathematics in financial planning that provide interesting results (Coopersmith & Sumutka, 2011; Schmidt, 2016; Deeley, 2007; Foziah, Ghazali, Mamat, Salleh, & Mohamed, 2017). This article presents the results of using mathematics on interest rates and present values to show that which retirement programs should be chosen? Each program is determined by the rate of returns under the same conditions and more importantly and it also enables us aware that saving for retirement requires monthly savings in each project.

Compound Interest

The easiest basic financial tool is to deposit with a financial institution or lend it with interest rates as the financial institution or borrower promises to pay. Interest rate characteristics can be classified as simple interest referred to the interests that are fixed on the principal throughout the terms of the deposit or loan. Compound interest is calculated by taking the interest of each period together with the principal of the next period in the future resulting in increased interest received in the next period, and continuously compounded interest means the actual compound interest rate that has an compounding amount per year approaching infinity. This article focuses on investments that include bringing interest to the principal to charge interest and call this type of investment that yields interest as compound interest.

A Total money

A_i Money earned from each interest rate cycle, $i = 1,2,...,m$

P Principal

r The annual interest rate

n Number of years

t The number of interest charges in one year

For the analysis processes, interest rates must be calculated per year, which is equal to $\frac{r}{n}$. The total interest payment cycle is multiplied between the numbers of interest charges per year t and the number of years n in which the value is nt so that $m = nt$. Money earned from each interest rate cycle is in the followings:

$$A_0 = P,$$
$$A_1 = A_0 + A_0 \tfrac{r}{n},$$
$$A_2 = A_1 + A_1 \tfrac{r}{n},$$
$$\vdots$$
$$A_m = A_{m-1} + A_{m-1} \tfrac{r}{n}.$$

From the equations of paying the compound interests of each round until the round m, we will get that,

$$A_m = A_{m-1} + A_{m-1} \tfrac{r}{n} = A_{m-1}\left(1+\tfrac{r}{n}\right),$$
$$A_m = A_{m-1}\left(1+\tfrac{r}{n}\right) = \left(A_{m-2} + A_{m-2}\tfrac{r}{n}\right)\left(1+\tfrac{r}{n}\right) = A_{m-2}\left(1+\tfrac{r}{n}\right)\left(1+\tfrac{r}{n}\right) = A_{m-2}\left(1+\tfrac{r}{n}\right)^2,$$
$$A_m = A_{m-2}\left(1+\tfrac{r}{n}\right)^2 = \left(A_{m-3} + A_{m-3}\tfrac{r}{n}\right)\left(1+\tfrac{r}{n}\right)^2 = A_{m-3}\left(1+\tfrac{r}{n}\right)\left(1+\tfrac{r}{n}\right)^2 = A_{m-3}\left(1+\tfrac{r}{n}\right)^3$$
$$\vdots$$
$$A_m = A_1\left(1+\tfrac{r}{n}\right)^{m-1} = \left(A_0 + A_0\tfrac{r}{n}\right)\left(1+\tfrac{r}{n}\right)^{m-1} = A_0\left(1+\tfrac{r}{n}\right)\left(1+\tfrac{r}{n}\right)^{m-1} = A_0\left(1+\tfrac{r}{n}\right)^m$$

$$A_m = P\left(1+\tfrac{r}{n}\right)^m.$$

Since $m = nt$, the formula of compound interest is $A = P\left(1+\tfrac{r}{n}\right)^{nt}$.

Present Value

The present value is the amount of money P invested in the current year with compound interest rates r per year in order to get the total money A in the future years or we can say that the present value P is the total devaluation A for a period of n years, where one year is given by interest t times with compound interest rates r per year and we will retrospectively receive the present value, which is $P = A\left(1+\tfrac{r}{n}\right)^{-nt}$. The present value can be thought of as the geometric series and so we can calculate the sum. Let S be the partial sum of such geometric series and $a_1, a_2, ..., a_n$ be the geometric sequences, $n \in Z$,

$$S = 1 + a + a^2 + ... + a^n.$$

By multiplying a to both sides of the equation, we obtain

$$aS = a + a^2 + a^3 + ... + a^{n+1}.$$

Subtracting the first equation by the second equation, we obtain

$$S - aS = \left(1 + a + a^2 + ... + a^n\right) - \left(a + a^2 + a^3 + ... + a^{n+1}\right)$$

$$S - aS = 1 - a^{n+1}$$

$$S = \frac{1-a^{n+1}}{1-a}, \quad a \neq 1.$$

This conceptual knowledge will be used to find the present value of all deposits in the saving accounts for retirements which retirees who save money can deposit money each month and then will find the present value of paying back all the money after retirement of a financial institution which retirees who save money in the project have to withdraw money each month. Set d to be the amount of savings each month. r is compound interest rate per year with interest charged a number of t times per year for a period of n years. When depositing fixed savings per month is d , the present value of all deposits in saving accounts for retirement can be obtained by combining each present value of savings from the times $1,2,..,nt$ which can be considered as follows:

$$d\sum_{i=1}^{nt}\left(1+\tfrac{r}{12}\right)^{-i} = d\left(1+\tfrac{r}{12}\right)^{-1} + d\left(1+\tfrac{r}{12}\right)^{-2} + ... + d\left(1+\tfrac{r}{12}\right)^{-nt}$$

$$= d\left(1+\tfrac{r}{12}\right)^{-1}\left[1 + \left(\left(1+\tfrac{r}{12}\right)^{-1}\right)^{1} + \left(\left(1+\tfrac{r}{12}\right)^{-1}\right)^{2} ... + \left(\left(1+\tfrac{r}{12}\right)^{-1}\right)^{nt-1}\right]$$

$$= d\left(1+\tfrac{r}{12}\right)^{-1}\left[\frac{1-\left(\left(1+\tfrac{r}{12}\right)^{-1}\right)^{nt-1+1}}{1-\left(1+\tfrac{r}{12}\right)^{-1}}\right]$$

$$= d\left(1+\tfrac{r}{12}\right)^{-1}\left[\frac{1-\left(1+\tfrac{r}{12}\right)^{-nt}}{1-\left(1+\tfrac{r}{12}\right)^{-1}}\right].$$

Therefore, the current value of all deposits in the retirement savings account is

$$d\sum_{i=1}^{nt}\left(1+\tfrac{r}{12}\right)^{-i}=d\left(1+\tfrac{r}{12}\right)^{-1}\left(\frac{1-\left(1+\tfrac{r}{12}\right)^{-nt}}{1-\left(1+\tfrac{r}{12}\right)^{-1}}\right).$$

Let s be the amount of money that the savings person wants to withdraw each month after retirement and it has started to be withdraw from the beginning of the month m as a compound interest rate per year with calculated interest t times per year. p is the total number of years from the beginning of the deposit until the last year that has been paid back after retirement and it can be written as the sequence of the total number of times as $1,2,...,nt,m,m+1,m+2,...,pt$. The present value of all annual nightly saving payments can be considered as $s\sum_{i=m}^{pt}\left(1+\tfrac{r}{12}\right)^{-i}$. Let $k=i-(m-1)$, so $i=m+(k-1)$ and $-i=-m-(k-1)$. Substituting $i=m$ gives $k=(m)-(m-1)=1$. Substituting $i=pt$ yields $k=(pt)-(m-1)=pt-nt$. Because $m-1$ is the same value as nt, therefore, the current value of all annual saving payments can be considered as follows:

$$\begin{aligned}s\sum_{i=m}^{pt}\left(1+\tfrac{r}{12}\right)^{-i}&=s\sum_{k=1}^{pt-nt}\left(1+\tfrac{r}{12}\right)^{-m-(k-1)}\\&=s\left(1+\tfrac{r}{12}\right)^{-m}\left[1+\left(\left(1+\tfrac{r}{12}\right)^{-1}\right)^{1}+\left(\left(1+\tfrac{r}{12}\right)^{-1}\right)^{2}+\ldots+\left(\left(1+\tfrac{r}{12}\right)^{-1}\right)^{(pt-nt-1)}\right]\\&=s\left(1+\tfrac{r}{12}\right)^{-m}\left[\frac{1-\left(\left(1+\tfrac{r}{12}\right)^{-1}\right)^{pt-nt-1+1}}{1-\left(1+\tfrac{r}{12}\right)^{-1}}\right]\\&=s\left(1+\tfrac{r}{12}\right)^{-m}\left[\frac{1-\left(1+\tfrac{r}{12}\right)^{-(pt-nt)}}{1-\left(1+\tfrac{r}{12}\right)^{-1}}\right].\end{aligned}$$

Therefore, the current value of all refunds in the retirement savings account is

$$s\sum_{i=m}^{pt}\left(1+\tfrac{r}{12}\right)^{-i}=s\left(1+\tfrac{r}{12}\right)^{-m}\left(\frac{1-\left(1+\tfrac{r}{12}\right)^{-(pt-nt)}}{1-\left(1+\tfrac{r}{12}\right)^{-1}}\right).$$

For the analysis process in this article, r is not considered directly as an interest rate but r is considered from the equation $P=A(1+r)^{-1}$ which corresponds to $r=\frac{A}{P}-1$. The value of r obtained from this formula is called the rate of return. Because the rate of return is equivalent to the interest rate, therefore r will be also used as the rate of return.

Another factor that should be considered for the analysis of the rate of return is about the rate of inflation. There are many causes of inflation. The rate of inflation affects planning for retirement savings. Let r_c be the interest rate of deposit and let r_f be the rate of inflation. The amount of deposited money reduces because of having deposit more with the actual interest rate r_c+r_f while the amount of money from saving for retirement after retirement has decreased because the money withdrawn has the value of money compared to the interest calculation with actual interest rate $r_c - r_f$.

Analysis the Rate of Return of Saving for Retirement

This section will demonstrate the rate of return by analyzing the present value from 4 real savings projects which are the Saving Project 1 Pro annuity A85/5, the Saving Project 2 Pro annuity A90/60, the Saving Project 3 Retire smart 1st 620, and the Saving Project 4 Retire smart 1st 620/60 under the same conditions to compare the differences and finally to choose the appropriate saving project.

r The rate of return

d Amount of savings each month

s Amount of money spent each month after retirement

m The first month of the initial withdrawal

w The first month of the initial deposit

p The total number of years from the beginning of the deposit until the last year that has been paid back after retirement

r_1 The rate of return of saving project 1 **Pro annuity A85/5**

r_2 The rate of return of saving project 2 **Pro annuity A90/60**

r_3 The rate of return of saving project 3 **Retire smart 1^{st} 620**

r_4 The rate of return of saving project 4 **Retire smart 1^{st} 620/60**

r_f The inflation rate of Thailand.

The rate of return for each project for retirement savings results from the formula $r = \frac{A}{P} - 1$ and the rate of inflation in Thailand is equal to $r_f = 1.5\% = 0.01500$.

1. The Saving Project 1 Pro annuity A85/5.

The rate of return used to find the amount of deposit each month is

$r_c + r_f = 0.01412 + 0.01500 = 0.02912$.

The rate of return used to find the amount of withdrawal each month is

$r_c - r_f = 0.01412 - 0.01500 = 0.00088$.

2. The Saving Project 2 Pro annuity A90/60.

The rate of return used to find the amount of deposit each month is

$r_c + r_f = 0.01320 + 0.01500 = 0.02820$.

The rate of return used to find the amount of withdrawal each month is

$r_c - r_f = 0.01320 - 0.01500 = -0.00180$.

3. The Saving Project 3 Retire smart 1st 620.

The rate of return used to find the amount of deposit each month is

$r_c + r_f = 0.00997 + 0.01500 = 0.02497$.

The rate of return used to find the amount of withdrawal each month is

$r_c - r_f = 0.00997 - 0.01500 = -0.00503$.

4. The Saving Project 4 Retire smart 1st 620/60.

The rate of return used to find the amount of deposit each month is

$r_c + r_f = 0.01385 + 0.01500 = 0.02885$.

The rate of return used to find the amount of withdrawal each month is

$r_c - r_f = 0.01385 - 0.01500 = -0.00115$.

Conditions for analysis of all four saving plans are peoples begins saving money at the age of 30 years and they are retired at the age of 60 years and wants to spend money until the age of 80 years, wanting to spend money 10,000 baht monthly. Therefore, it takes 30 years to save money which is equal to $12 \times 30 = 360$ months. Therefore, $nt = 360$. The time to spend 20 years after retirement, which is equal to $12 \times 20 = 240$ months.

1. The Saving Project 1 Pro annuity A85/5.

The present value of all deposits in a saving account for retirement is

$$d\sum_{i=1}^{360}\left(1+\tfrac{0.02912}{12}\right)^{-i} = d\left(1+\tfrac{0.02912}{12}\right)^{-1}\left(\frac{1-\left(1+\tfrac{0.02912}{12}\right)^{-360}}{1-\left(1+\tfrac{0.02912}{12}\right)^{-1}}\right)$$

$$\approx 239.88130d .$$

The present value of all refunds in a saving account for retirement is

$$10000\sum_{i=m}^{pt}\left(1+\tfrac{-0.00088}{12}\right)^{-i} = s\left(1+\tfrac{-0.00088}{12}\right)^{-361}\left(\frac{1-\left(1+\tfrac{-0.00088}{12}\right)^{-240}}{1-\left(1+\tfrac{-0.00088}{12}\right)^{-1}}\right)$$

$$\approx 2{,}486{,}110.90400 .$$

So, $239.88130d \approx 2486110.90400$

$d \approx 10363.921$.

Therefore, amount of save money each month is $10{,}364$ bath.

2. The Saving Project 2 Pro annuity A90/60.

The present value of all deposits in a savings account for retirement is

$$d\sum_{i=1}^{360}\left(1+\tfrac{0.02820}{12}\right)^{-i} = d\left(1+\tfrac{0.02820}{12}\right)^{-1}\left(\frac{1-\left(1+\tfrac{0.02820}{12}\right)^{-360}}{1-\left(1+\tfrac{0.02820}{12}\right)^{-1}}\right)$$

$$\approx 242.74290d .$$

The present value of all refunds in saving accounts for retirement is

$$10000\sum_{i=m}^{pt}\left(1+\tfrac{-0.00180}{12}\right)^{-i} = s\left(1+\tfrac{-0.00180}{12}\right)^{-361}\left(\frac{1-\left(1+\tfrac{-0.00180}{12}\right)^{-240}}{1-\left(1+\tfrac{-0.00180}{12}\right)^{-1}}\right)$$

$$\approx 2{,}579{,}519.52300 .$$

So, $242.74290d \approx 2579519.52300$

$d \approx 10626.549$.

Therefore, amount of save money each month is $10{,}627$ bath.

3. The Saving Project 3 Retire smart 1st 620.

The present value of all deposits in a savings account for retirement is

$$d\sum_{i=1}^{360}\left(1+\tfrac{0.02497}{12}\right)^{-i} = d\left(1+\tfrac{0.02497}{12}\right)^{-1}\left(\frac{1-\left(1+\tfrac{0.02497}{12}\right)^{-360}}{1-\left(1+\tfrac{0.02497}{12}\right)^{-1}}\right)$$

$$\approx 253.18700d\ .$$

The present value of all refunds in saving accounts for retirement is

$$10000\sum_{i=m}^{pt}\left(1+\tfrac{-0.00503}{12}\right)^{-i} = s\left(1+\tfrac{-0.00503}{12}\right)^{-361}\left(\frac{1-\left(1+\tfrac{-0.00503}{12}\right)^{-240}}{1-\left(1+\tfrac{-0.00503}{12}\right)^{-1}}\right)$$

$$\approx 2{,}936{,}864.04300\,.$$

So, $253.18700d \approx 2936864.04300$

$$d \approx 11599.580\,.$$

Therefore, amount of save money each month is $11{,}600$ bath.

4. The Saving Project 4 Retire smart 1^{st} 620/60.

The present value of all deposits in a saving account for retirement is

$$d\sum_{i=1}^{360}\left(1+\tfrac{0.02885}{12}\right)^{-i} = d\left(1+\tfrac{0.02885}{12}\right)^{-1}\left(\frac{1-\left(1+\tfrac{0.02885}{12}\right)^{-360}}{1-\left(1+\tfrac{0.02885}{12}\right)^{-1}}\right)$$

$$\approx 240.71610d\ .$$

The present value of all refunds in savings accounts for retirement is

$$10000\sum_{i=m}^{pt}\left(1+\tfrac{-0.00115}{12}\right)^{-i} = s\left(1+\tfrac{-0.00115}{12}\right)^{-361}\left(\frac{1-\left(1+\tfrac{-0.00115}{12}\right)^{-240}}{1-\left(1+\tfrac{-0.00115}{12}\right)^{-1}}\right)$$

$$\approx 2{,}513{,}159.94600\,.$$

So, $240.71610d \approx 2513159.94600$

$$d \approx 10440.349$$

Therefore, amount of save money each month is $10,440$ bath.

Results of the amount of savings in each month of the saving project for retirement project can be shown in the table as follows (Table 1).

Table 1. The amount of savings in each month of the saving project for retirement project

Projects	Name of projects	Amount of savings per month (unit : bath)
1	Pro annuity A85/5	10,364
2	Pro annuity A90/60	10,627
3	Retire smart 1^{st} 620	11,600
4	Retire smart 1^{st} 620/60	10,440

From the above table, under the same conditions, the first project which is Pro annuity A85/5 has the lowest amount of saving money which is 10,364 baht which is the best project to be chosen. If in real life there are conditions according to the analysis, it will know which project to save for retirement. If the actual conditions are different, they should consider additional conditions for making investment decisions.

Summary

From the mathematical analysis of returns using present value analysis enables people them aware that the most savings should be invested with the saving project. This analysis also helps them know how to plan for retirement savings in order to get the most benefits. We also know that the more investment, the faster saving and doing this benefits more than the slow savings investment, and also note that in addition to the rate of return, it must also take into account the duration of savings, which will also have inflation rate. All mentioned here will make the savings plan to be achieved the goal for a better quality of life for the future.

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