บทความวิชาการ

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บทคัดย่อ

การประมาณค่าอินทิกรัลจำกัดเขตเป็นเรื่องที่นักคณิตศาสตร์ให้ความสนใจศึกษาเรื่องหนึ่ง โดยเฉพาะอย่างยิ่งสำหรับฟังก์ชันที่หาค่าอินทิกรัลเชิงวิเคราะห์ไม่ได้ ซึ่งมีการเสนอหลายวิธีเพื่อใช้ในการประมาณค่าอินทิกรัลจำกัดเขต วิธีการประมาณอินทิกรัลจำกัดเขตวิธีการหนึ่งที่ให้ผลลัพธ์เป็นที่น่าพอใจ คือ การประมาณฟังก์ชันที่อินทิกรัลเชิงวิเคราะห์ไม่ได้ด้วยฟังก์ชันที่อินทิกรัลเชิงวิเคราะห์ได้ ถ้าฟังก์ชันประมาณเป็นฟังก์ชันที่ดีที่สามารถแทนฟังก์ชันที่ต้องการพิจารณาอินทิกรัลจำกัดเขต แล้วอินทิกรัลจำกัดเขตของฟังก์ชันประมาณก็จะเป็นค่าประมาณที่ดีที่สามารถแทนอินทิกรัลจำกัดเขตของฟังก์ชันที่ต้องการพิจารณาได้ บทความนี้มีจุดประสงค์เพื่อศึกษาการประมาณฟังก์ชันด้วยการประมาณค่าในช่วงโดยใช้สไปลน์กำลังสาม และเพื่อประยุกต์ฟังก์ชันประมาณที่ได้จากการประมาณค่าในช่วงโดยใช้สไปลน์กำลังสาม เพื่อประมาณอินทิกรัลจำกัดเขตของฟังก์ชัน ในบทความได้นำเสนอการประมาณฟังก์ชันด้วยการประมาณค่าในช่วงโดยใช้สไปลน์กำลังสาม ทั้งฟังก์ชันที่หาค่าอินทิกรัลเชิงวิเคราะห์ได้และหาค่าอินทิกรัลเชิงวิเคราะห์ไม่ได้ เพื่อให้เห็นชัดเจนถึงวิธีการประยุกต์สไปลน์กำลังสามเพื่อประมาณค่าอินทิกรัลจำกัดเขต

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Academic Article

Applying cubic spline to estimate definite integrals

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Abstract

Evaluating definite integral is of mathematician interests, especially applying with nonintegrable functions. There are various ways to do so and one of them is to evaluate a non-analytic integrable function by using an analytic integrable function. If an estimated function is fairly good, we can replace the evaluating definite integral by the actual definite integral. This work aims to evaluating estimated functions by cubic spline method, its applications and definite integrals of the estimated functions. We also present evaluating functions by estimating them in intervals with cubic spline in both analytic integrable and non-analytic integrable functions to focus clearly on the applications of using cubic spline for definite integrals.

Keywords: interval estimation, cubic spline, definite integral estimation

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Definite Integral and Interval Estimation

Definite integral plays an important role in mathematics and it has applications in various ways (Heinbockel, 2012; Bittinger, Ellenbogen and Surgent, 2015). There is, however, drawback application for the usage of definite integral which is we cannot apply this method to some functions that are non-analytic integrable and hence the actual definite integral cannot be evaluated. So such definite integral has been estimated which was presented earlier (Eldén and Wittmeyer-Koch, 1990; Burden and Faires, 2001; Steven, 2007). One of the good estimations of the definite integral is estimation non-analytic integrable functions by using analytic integrable functions. If the estimated function is good enough so that we can replace a considered function by such an estimated function, then the definite integral of an estimated function is also a good estimation for the considered function.

Estimation of non-analytic integrable functions can be evaluated on a considered interval by using polynomials called polynomial interpolation. To get more accurate in the estimation, higher degree polynomials are needed. However, the higher degrees of polynomials are added, the more errors of estimation occur at the same time. Therefore, the spline interpolation is introduced and its procedure is to divide a closed interval into finite subintervals and then there is a polynomial in each subinterval (Kawamura, Sasaki and Otsuki, 1992; McKinley and Levine, 1998; Giordano, Fox and Horton, 2013). This method will connect piecewise smooth curves and we call this a linear spline interpolation which is the simplest type of spline interpolation but it is not widely used. However, cubic spline interpolation is efficient method due to the continuity of second-order differentiation. This results in a good estimation of a considered function by using the cubic spline interpolation (Burt and Adelson, 1983; Wang, Shen and Teoh, 2000; Kastanek and Nielsen, 2001; Scardapane, Scarpiniti, Comminiello and Uncini, 2017).

This work aims to apply an estimated function from cubic spline interpolation to find a definite integral. We present cubic spline interpolation in both analytic integrable and non-analytic integrable functions to see an application of cubic spline for finding definite integrals.

Cubic spline interpolation

Using polynomials to approximate some functions is of mathematician interests because polynomials have nice properties, namely, integrability and differentiability. The higher degree of polynomials, however, they tend to have more oscillators near the endpoints of the interval than the lower degree ones and then the stability of the approximation cannot be guaranteed properly due to the change of its coefficients. One of the reasons that the lower degree polynomials cannot be used in this method is an approximation of the curve with lower degree polynomials is not fitted properly for some data on the considered intervals, especially, non-quadratic or cubic data.

We present, in this topic, modern and widely used techniques called cubic spline interpolation which uses cubic polynomials to connect in each subinterval.

We present here the cubic spline interpolation from four points of data of a function f : $(x_1, f(x_1))$, $(x_2, f(x_2))$, $(x_3, f(x_3))$ and $(x_4, f(x_4))$

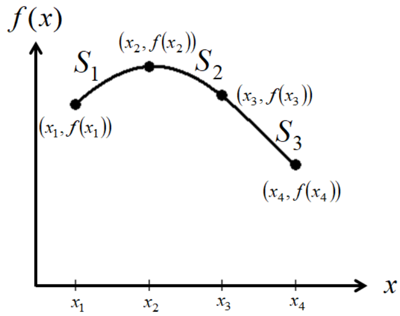

Figure 1. Cubic spline interpolation from four data points

We denote

$$S_1(x) = a_1 + b_1x + c_1x^2 + d_1x^3, x \in [x_1, x_2)$$
$$S_2(x) = a_2 + b_2x + c_2x^2 + d_2x^3, x \in [x_2, x_3)$$
$$S_3(x) = a_3 + b_3x + c_3x^2 + d_3x^3, x \in [x_3, x_4].$$

Condition 1: continuity at points.

Conditions	Equations	Order
$S_1(x_1) = f(x_1)$	$a_1 + b_1x_1 + c_1x_1^2 + d_1x_1^3 = f(x_1)$	(1)
$S_1(x_2) = f(x_2)$	$a_1 + b_1x_2 + c_1x_2^2 + d_1x_2^3 = f(x_2)$	(2)
$S_2(x_2) = f(x_2)$	$a_2 + b_2x_2 + c_2x_2^2 + d_2x_2^3 = f(x_2)$	(3)
$S_2(x_3) = f(x_3)$	$a_2 + b_2x_3 + c_2x_3^2 + d_2x_3^3 = f(x_3)$	(4)
$S_3(x_3) = f(x_3)$	$a_3 + b_3x_3 + c_3x_3^2 + d_3x_3^3 = f(x_3)$	(5)
$S_3(x_4) = f(x_4)$	$a_3 + b_3x_4 + c_3x_4^2 + d_3x_4^3 = f(x_4)$	(6)

Condition 2: continuity of slopes.

Conditions	Equations	Order
$S_1'(x_2) = S_2'(x_2)$	$b_1 + 2c_1x_2 + 3d_1x_2^2 - b_2 - 2c_2x_2 - 3d_2x_2^2 = 0$	(7)
$S_2'(x_3) = S_3'(x_3)$	$b_2 + 2c_2x_3 + 3d_2x_3^2 - b_3 - 2c_3x_3 - 3d_3x_3^2 = 0$	(8)

Condition 3: continuity of 2nd derivatives.

Conditions	Equations	Order
$S_1''(x_2) = S_2''(x_2)$	$2c_1 + 6d_1x_2 - 2c_2 - 6d_2x_2 = 0$	(9)
$S_2''(x_3) = S_3''(x_3)$	$2c_2 + 6d_2x_3 - 2c_3 - 6d_3x_3 = 0$	(10)

Condition 4: Known derivative values at the extremities called clamped splines.

Conditions	Equations	Order
$S_1'(x_1) = f'(x_1)$	$b_1 + 2c_1x_1 + 3d_1x_1^2 = f'(x_1)$	(11)
$S_3'(x_4) = f'(x_4)$	$b_3 + 2c_3x_4 + 3d_3x_4^2 = f'(x_4)$	(12)

Equations (1)-(12) will form linear systems which give coefficients $S_1, S_2,$ and S_3 and they are used to approximate the function $f(x)$ in the closed interval $[x_1, x_4]$.

Cubic spline interpolation for elementary functions

This section demonstrates the estimation of elementary functions using cubic spline interpolation and these demonstrations lead to the approximation of the definite integrals of these functions. We then are able to compare the actual and the approximation values by using cubic spline and so we can see applications of the cubic spline interpolation in the definite integrals of elementary functions on their domains as shown in the below table.

Functions	Domains of Functions
$f(x) = x^4$	$[0,1]$
$f(x) = \sin x$	$[0, \pi/2]$
$f(x) = e^x$	$[0,1]$
$f(x) = \ln x$	$[2,4]$

By the method of cubic spline interpolation, we divide a domain into 3 equally subintervals and this result in four points in the domain and we then obtain an estimated formula that will be used to substitute the actual function. We provide the graphs in both approximated and actual functions on domains as follows:

The approximated formula of $f(x) = x^4$ on $[0,1]$ is

$$x^4 \approx \begin{cases} -0.1111x^2 + 0.6667x^3; & 0 \le x < 1/3 \\ -0.0494 + 0.4444x - 1.4444x^2 + 2.0000x^3; & 1/3 \le x < 2/3 \\ -0.4444 + 2.2222x - 4.1111x^2 + 3.3333x^3; & 2/3 \le x \le 1. \end{cases}$$

Graphs of approximated and actual functions on $[0,1]$ are provided as below:

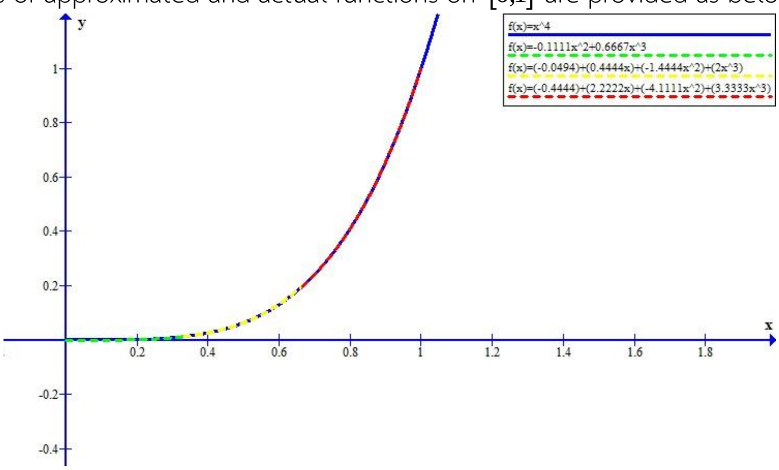

Figure 2. Graphs of $f(x) = x^4$ and approximated functions by using cubic spline interpolation

The approximated formula of $f(x) = \sin x$ on $[0, \pi/2]$ is

$$\sin x \approx \begin{cases} 1.0000x - 0.0651x^2 - 0.0401x^3; & 0 \le x < \pi/6 \\ 0.0638 + 0.6347x + 0.6326x^2 - 0.4843x^3; & \pi/6 \le x < \pi/3 \\ -1.9783 + 6.4847x - 4.9537x^2 + 1.2939x^3; & \pi/3 \le x \le \pi/2. \end{cases}$$

Graphs of approximated and actual functions on $[0, \pi/2]$ are provided as below:

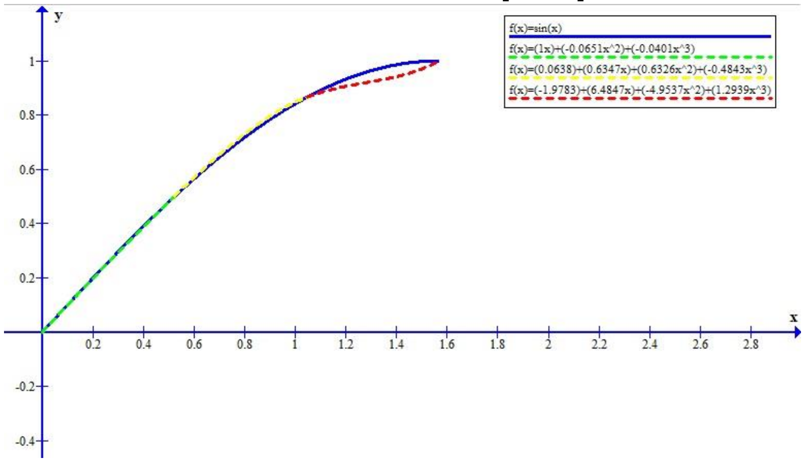

Figure 3. Graphs of $f(x) = \sin x$ and approximated functions by using cubic spline interpolation

The approximated formula of $f(x) = e^x$ on $[0,1]$ is

$$e^x \approx \begin{cases} 1.0000 + 1.0000x + 0.4950x^2 + 0.1965x^3; & 0 \le x < 1/3 \\ 0.9972 + 1.0255x + 0.4185x^2 + 0.2731x^3; & 1/3 \le x < 2/3 \\ 0.9646 + 1.1721x + 0.1986x^2 + 0.3830x^3; & 2/3 \le x \le 1. \end{cases}$$

Graphs of approximated and actual functions on $[0,1]$ are provided as below:

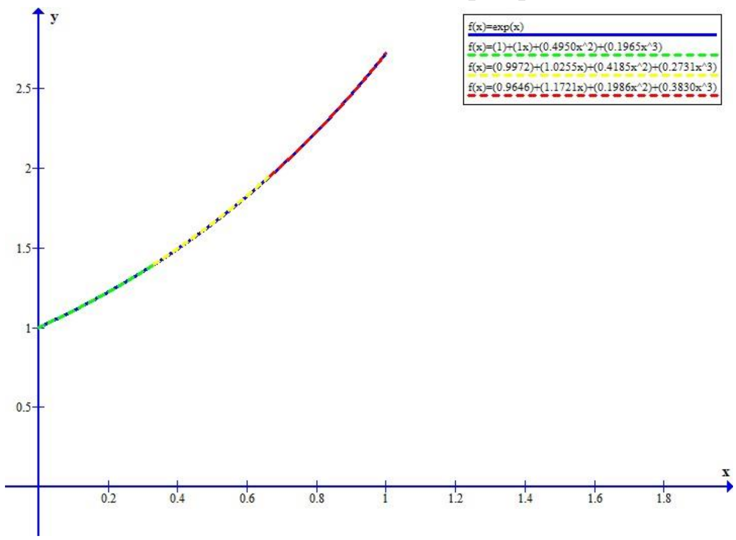

Figure 4. graphs of $f(x) = e^x$ and approximated functions by using cubic spline interpolation

The approximated formula of $f(x) = \ln x$ on $[2,4]$ is

$$\ln x \approx \begin{cases} -0.6103 + 0.8236x - 0.0884x^2 + 0.0013x^3; & 2 \le x < 8/3 \\ -0.9385 + 1.1928x - 0.2269x^2 + 0.0186x^3; & 8/3 \le x < 10/3 \\ -0.4179 + 0.7243x - 0.0864x^2 + 0.0045x^3; & 10/3 \le x \le 4. \end{cases}$$

Graphs of approximated and actual functions on $[2,4]$ are provided as below:

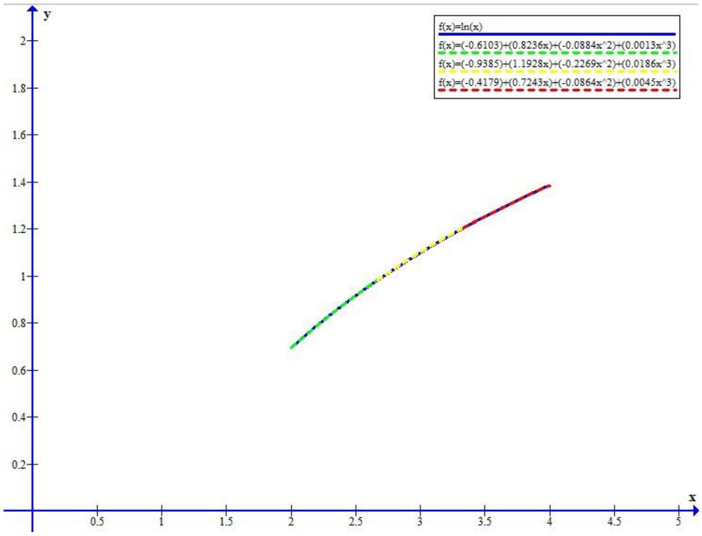

Figure 5. Graph of $f(x)=\ln x$ and approximated functions by using cubic spline interpolation

From the above applications of definite integral results, we can conclude in the following table:

Functions	Domains	Actual values of definite integrals	Approximated values of definite integrals	Relative errors
$f(x)=x^4$	$[0,1]$	0.2000	0.1996	0.2004
$f(x)=\sin x$	$[0,\pi/2]$	1.0000	0.9885	1.1634
$f(x)=e^x$	$[0,1]$	1.7183	1.7182	0.0058
$f(x)=\ln x$	$[2,4]$	2.1589	2.1589	0.0000

We can observe from the table that the relative errors are very small so the results are reasonable and can be applied. In the next section, we present the application of definite integrals using cubic spline interpolation for non-analytic integrable functions.

Cubic spline interpolation for non-analytic integrable functions.

We demonstrate how to make uses of the results from the previous section in order to approximate the definite integrals of non-analytic integrable functions using cubic spline interpolation. We give two examples of non-analytic integrable functions which cannot use other methods easily to do so.

The first one is the approximation formula of $f(x)=e^{x^2}$ on $[0,1]$ which is

$$e^{x^2} \approx \begin{cases} 1.0000 + 0.9378x^2 + 0.3597x^3; & 0 \le x < 1/3 \\ 0.9637 + 0.3264x - 0.0414x^2 + 1.3388x^3; & 1/3 \le x < 2/3 \\ -0.0820 + 5.0323x - 7.1003x^2 + 4.8683x^3; & 2/3 \le x \le 1. \end{cases}$$

Graphs of approximated and actual functions on $[0,1]$ are provided as below:

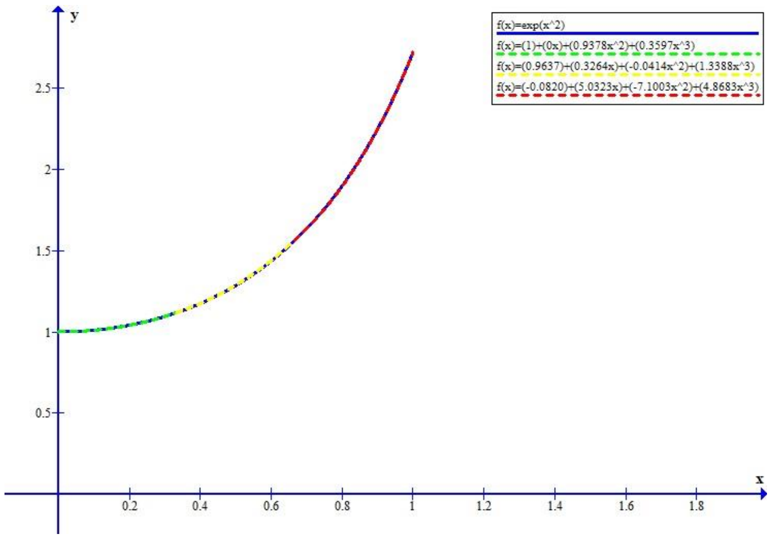

Figure 6. Graphs of $f(x) = e^{x^2}$ and approximated functions by using cubic spline interpolation

The approximated value of definite integrals is about 1.4617 compared to using mathematics software which gives 1.4627 and we consider these two values are closed.

The second function is approximation formula of $f(x) = \cos(x^2)$ on $[0,1]$ which is

$$\cos\left(x^2\right) \approx \begin{cases} 1.0000 + 0.0532x^2 - 0.3262x^3; & 0 \le x < 1/3 \\ 1.0242 - 0.2175x + 0.7057x^2 - 0.9786x^3; & 1/3 \le x < 2/3 \\ 0.9717 + 0.0186x + 0.3515x^2 - 0.8015x^3; & 2/3 \le x \le 1. \end{cases}$$

Graphs of approximated and actual functions on $[0,1]$ are provided as below:

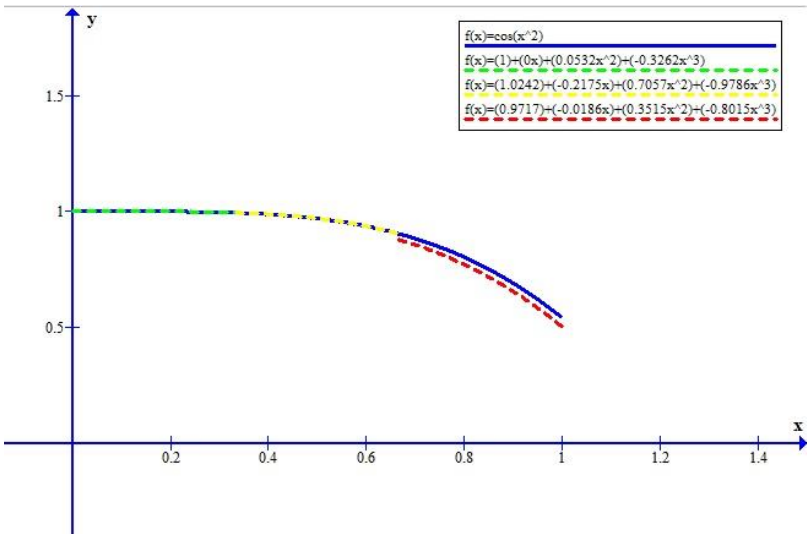

Figure 7. Graphs of $f\left(x\right)=\cos(x^2)$ and approximated functions by using cubic spline interpolation

An approximated value of the definite integral is about 0.9045 compared to using mathematics software which gives 0.8942 and we consider these two values are closed.

Summary

Using cubic spline interpolation in an application of definite integrals can be used for both non-analytic and analytic integrable functions. The results show six different functions in which the first four are analytic integrable functions and here are elementary functions and the last two functions are non-analytic integrable functions. The actual and the approximation values yield acceptable small errors so that we can use the approximated values instead of the actual ones. For further studies or other sophisticated non-analytic integrable functions or analytic integrable can be experimented.

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