บทความวิจัย

ผลเฉลยที่ไม่เป็นจำนวนเต็มลบของสมการไดโอแฟนไทน์ $15^x + 51^y = z^2$

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บทคัดย่อ

ในบทความนี้ เราได้ศึกษาผลเฉลยที่ไม่เป็นจำนวนเต็มลบของสมการไดโอแฟนไทน์ $15^x + 51^y = z^2$ โดยที่ x, y และ z เป็นจำนวนเต็มที่ไม่เป็นลบ พบว่า คำตอบของสมการไดโอแฟนไทน์มีผลเฉลยที่ไม่เป็นจำนวนเต็มลบเพียงผลเฉลยเดียว คือ (x, y, z) = (1, 0, 4)

คำสำคัญ: สมการไดโอแฟนไทน์เอกโพเนนเชียล ข้อคาดการณ์ของคาตาลัน

อ้างอิงบทความนี้
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Research Article

The non-negative integer solutions of Diophantine equation $15^x + 51^y = z^2$

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Abstract

In this paper, we study non-negative integer solutions of Diophantine equation $15^x + 51^y = z^2$ where x, y and z are non-negative integers. We show that the Diophantine equation has only one solution (x, y, z) = (1, 0, 4) in non-negative integers.

Keywords: Exponential Diophantine equation, Catalan's conjecture

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Introduction

The Diophantine equations of type $a^x + b^y = c^z$ have been studied (Acu, 2005; Acu, 2007; Suvarnamani, Singta andChotchaisthit, 2011; Sroysang, 2012a; Sroysang, 2012b; Sroysang, 2012c; Sroysang, 2013; Rabago, 2013a; Rabago, 2013b; Rabago, 2013c; Rabago, 2013d; Rabago, 2013e). In 2005 (Acu, 2005), Acu studied Diophantine equations of type $a^x + b^y = c^z$ for primes a and b. In 2007 (Acu, 2007), Acu studied Diophantine equations $2^x + 5^y = z^2$. In 2008 (Pumnea and Nicoar; 2008), Pumnea et al. studied Diophantine equations of the form $a^x + b^y = z^2$, for example: $2^x + 7^y = z^2$, $2^x + 11^y = z^2$ and $2^x + 13^y = z^2$ and. In 2011 (Suvarnamani, 2011), Suvarnamani studied Diophantine equation $2^x + p^y = z^2$ where p is prime number which more than 2, he found that (i) For each prime number p, this equation has a solution (x, y, z) = (3, 0, 3), (ii) For p = 3, the Diophantine equation this equation has a solution (x, y, z) = (4, 2, 5) and (iii) For $p = 1 + 2^{k+1}$ where k is non-negative integer, this equation has a solution (x, y, z) = (2k, 1, 1+2k). In 2011 (Suvarnamani, Singta and Chotchaisthit, 2011), Suvarnamani et al. studied Diophantine two equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$. In 2012-2013, Sroysang published series of papers in relation to the Diophantine equation $a^x + b^y = c^z$ (Sroysang, 2012a; Sroysang, 2012; Sroysang, 2012c; Sroysang, 2013). In 2013 (Rabago, 2013c), Rabago gave all solutions to several Diophantine equations of type $p^x + q^y = z^2$, for examples: $5^x + 31^y = z^2$, $7^x + 29^y = z^2$, $13^x + 23^y = z^2$, $47^x + 97^y = z^2$ and $61^x + 83^y = z^2$. In 2013 (Rabago, 2013d), Rabago studied the two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$. In the same year (Rabago, 2013e), Rabago studied the two Diophantine equations $17^x + 19^y = z^2$ and $71^x + 73^y = z^2$. In most of these papers, the authors used theory of congruence and/or Catalan's conjecture to find all solutions, or to show the non-existence of solutions to the Diophantine equations of type $p^x + q^y = z^2$. In 2014 (Sroysang, 2014), Sroysang studied the Diophantine equation $46^x + 64^y = z^2$, he used theory of congruence and/or Catalan's conjecture to find all solutions, or to show the non-existence of solutions and he found that this equation has no non-negative integers solution. Furthermore, he posed some open problem about the Diophantine equation in the form $m^x + n^y = z^2$ where m = 10a +b, n = 10b + a with a, b are in {0, 1, 2, ..., 9}. We see that there are many Diophantine equations in the previous form. In 2021 (Ngarm-pong, Raumtum and Thongmoon, 2021), Ngarm-pong et al. studied the Diophantine equation $13^x + 31^y = z^2$, they found that this equation has no solution in non-negative integers.

In this paper, we study the Diophantine equation

$$15^x + 51^y = z^2 \qquad \text{........ (1)}$$

Where x, y and z are non-negative integers. This equation is one type of Sroysang's open problem.

Preliminaries

Proposition 2.1 (Catalan's conjecture**)** (Mihailescu, 2004) (3, 2, 2, 3) is a unique solution (a, b, x, y) for the Diophantine equation $a^x – b^y = 1$ where a, b, x and y are non-negative integers with min{a, b, x, y} > 1.

Main Results

Lemma 3.1 The Diophantine equation $15^x + 1 = z^2$ has only one solution (x, z) = (1, 4) in non-negative integer.

Proof. Let x and z be non-negative integers. Clearly, when x = 0 and z = 0. Then we will consider in case x > 0 by divided into two cases:

Case 1. If x = 1, then Diophantine equation $15^x + 1 = z^2$ becomes $z^2 = 16$. So z = 4, then we get that (x, z) = (1, 4) is a non-negative integers solution of Diophantine equation $15^x + 1 = z^2$.

Case 2. If x > 1, We see that $15^x = z^2 – 1 = (z – 1)(z + 1)$. Assume that $z – 1 = 15^u$ and $z + 1 = 15^{x-u}$ where u is non-negative integer and x > 2u. Then

$$15^{x-u} – 15^u = (z + 1) - (z – 1) = 2.$$

We see that

$$15^u(15^{x – 2u} - 1) = 2.$$

We will consider two subcases:

Subcase 2.1 If $15^u = 1$ and $15^{x-2u} - 1 = 2$, we imply that $u = 0$ and $15^x - 1 = 2$. We have that $15^x = 3$, this is impossible.

Subcase 2.2 If $15^u = 2$ and $15^{x-2u} - 1 = 1$. This is impossible.

From Case 1. and Case 2., the Diophantine equation $15^x + 1 = z^2$ has only one solution $(x, z) = (1, 4)$ in non-negative integers.

Lemma 3.2 The Diophantine equation $51^x + 1 = z^2$ has no solution in non-negative integer.

Proof. Let x and z be non-negative integers. Clearly, when $x = 0$ and $z = 0$. From $51^x + 1 = z^2$ we see that $z^2 - 51^x = 1$. By Catalan's conjecture, this equation has no non-negative solution when $\min\{z, 2, 51, x\} > 1$. Since x and z are non-negative integers. We will consider two cases as follows:

Case 1. If $z = 1$, then we get $51^x = 0$ which has no non-negative solution.

Case 2. If $x = 1$, then we get $z^2 - 51^x = 1$. This imply that $z^2 = 52$ which has no non-negative solution. Therefore, the Diophantine equation $51^x + 1 = z^2$ has no solution in non-negative integer.

Theorem 3.3 The Diophantine equation (1) has only one solution $(x, y, z) = (1, 0, 4)$ in non-negative integers.

Proof. Let x, y and z be non-negative integers. Clearly, if $z = 0$, then the Diophantine equation $15^x + 51^y = 0$ has no solution (x, y) in non-negative integers. Now, let $z > 0$ and we will consider in three cases:

Case 1. If $x = 0$, then the Diophantine equation (1) can be written as $1 + 51^y = z^2$. By Lemma 3.2, this equation has no non-negative integer solutions.

Case 2. If $y = 0$, then the Diophantine equation (1) can be written as $15^x + 1 = z^2$. By Lemma 3.1, this has only one solution $(x, z) = (1, 4)$ in non-negative integer. Therefore $(x, y, z) = (1, 0, 4)$ is a non-negative integer solution of the Diophantine equation (1).

Case 3. If $xy > 0$, then we will consider in two subcases:

Subcase 3.1 If x is even, assume that $x = 2k$ for some non-negative integer k. Then the Diophantine equation (1) can be written as

$$15^{2k} + 51^y = z^2$$

i.e.,

$$51^y = z^2 - 15^{2k} = (z-15^k)(z+15^k)$$

Assume that $z - 15^k = 51^u$ and $z + 15^k = 51^{y-u}$ where u is non-negative integer and $y > 2u$. Then

$$51^{y-u} - 51^u = 2(15^k).$$

We see that

$$51^u(51^{y-2u} - 1) = 2(15^k).$$

We will consider in four subcases:

Subcase 3.1.1 If $51^u = 1$ and $51^{y-2u} = 2(15^k)$, we imply that $u = 0$ and $51^y - 1 = 2(15^k)$. We have that $50(51^{y-1} + 51^{y-2} + \ldots + 51 + 1) = 2(15^k)$. Then we get $25t = 15^k$ where $t = 51^{y-1} + 51^{y-2} + \ldots + 51 + 1$. We see that $t \equiv 1 \pmod 3$. Then $5^2t = 3^k5^k$. This is impossible because $5^2t\ 1 \pmod 3$ but $3^k5^k \equiv 0 \pmod 3$.

Subcase 3.1.2 If $51^u = 2$ and $51^{y-2u} - 1 = 15^k$, this is impossible.

Subcase 3.1.3 If $51^u = 15^k$ and $51^{y-2u} - 1 = 2$, this is impossible.

Subcase 3.1.4 If $51^u = 2(15^k)$ and $51^{y-2u} - 1 = 1$, this is impossible.

Subcase 3.2 If x is odd, then $x = 2k + 1$ for some non-negative integer k. Then the Diophantine equation (1) becomes:

$$15^{2k+1} + 51^y = z^2$$

we will consider in two subcases:

Subcase 3.2.1 If y is even, then $y = 2m$ for some positive integer m.

Then the Diophantine equation (1) becomes:

$$15^{2k+1} + 51^{2m} = z^2$$

i.e.,

$$15^{2k+1} = z^2 - 51^{2m} = (z - 51^m)(z + 51^m).$$

We will consider in four subcases:

Subcase 3.2.1.1 If $z - 51^m = 1$ and $z + 51^m = 15^{2k+1}$, then $15^{2k+1} - 1 = 2(51^m)$. We have that $14(15^{2k} + 152k\text{-}1 + ... + 15 + 1) = 2(51^m)$. Then we get $7t = 51^m$ Where $t = 15^{2k} + 15^{2k-1} + ... + 15 + 1$. Since $7t \equiv 0 \pmod 7$, then $51^m \equiv 0 \pmod 7$. Contradiction to $51^m \equiv 1, 2, 4 \pmod 7$.

Subcase 3.2.1.2 If $z - 51^m = 15$ and $z + 51^m = 15^{2k}$, then $15^{2k} - 15 = 2(51^m)$. We have that $15(15^{2k-1} - 1) = 2(51^m)$. Since $15(15^{2k-1} - 1) \equiv 0 \pmod 5$, then $2(51^m) \equiv 0 \pmod 5$. Contradiction to $2(51^m) \equiv 2 \pmod 5$.

Subcase 3.2.1.3 If $z - 51^m = 15^{2k}$ and $z + 51^m = 15$, then $15 - 15^{2k} = 2(51^m)$. We have that $15(1 - 15^{2k-1}) = 2(51^m)$. Since $15(1 - 15^{2k-1}) \equiv 0 \pmod 5$, then $2(51^m) \equiv 0 \pmod 5$. Contradiction to $2(51^m) \equiv 2 \pmod 5$.

Subcase 3.2.1.4 If $z - 51^m = 15^{2k+1}$ and $z + 51^m = 1$, then $1 - 15^{2k+1} = z + 51^m = 2(51^m)$. This is impossible because that LHS is negative integer, but RHS is positive integer.

Subcase 3.2.2 If y is odd, then $y = 2m + 1$ for some non-negative integer m. Then the Diophantine equation (1) becomes: $15^{2k+1} + 51^{2m+1} = z^2$ we see that z^2 and z are also evens. Then z is either in the form 4s or 4s+2 for some non-negative integer s. So we will consider in two subcases:

Subcase 3.2.2.1 If $z = 4s$, then we get that $15^{2k+1} + 51^{2m+1} = 16s^2$. Since $15^{2k+1} + 51^{2m+1} \equiv 2 \pmod 4$, then $16s^2 \equiv 2 \pmod 4$. Contradiction to $16s^2 \equiv 0 \pmod 4$.

Subcase 3.2.2.2 If $z = 4s+2$, then we get that $15^{2k+1} + 51^{2m+1} = 16s^2 + 16s + 4$. Since $15^{2k+1} + 51^{2m+1} \equiv 2 \pmod 4$, then $16s^2 + 16s + 4 \equiv 2 \pmod 4$. Contradiction to $16s^2 + 16s + 4 \equiv 0 \pmod 4$.

From Case 1.,2. and Case 3., we have that the Diophantine equation (1) has only one solution (x, y, z) = (1, 0, 4) in non-negative integers. The proof is completed.

Theorem 3.4 Let $n \geq 2$ be a positive integer. Then the Diophantine equation $15^x + 51^y = w^{2n}$ has only one solution (x, y, w, n) = (1, 0, 2, 2) in non-negative integers.

Proof. Let $n \geq 2$ be a positive integer and w, x, y be non-negative integers. Assume that $z = w^n$, then the Diophantine equation $15^x + 51^y = w^{2n} = z^2$. By Theorem 3.3, we see that $z = 4$. Then $w^n = 4$ and we see that $w = 4$ and $n = 1$ or $w = 2$ and $n = 2$. By assumption $n \geq 2$, then we get that $w = 2$ and $n = 2$. Thus, the Diophantine equation $15^x + 51^y = w^{2n}$ has only one solution (x, y, w, n) = (1, 0, 2, 2) in non-negative integers.

Conclusion

In this paper, we show that the Diophantine equation $15^x + 51^y = z^2$ has only one solution (x, y, z) = (1, 0, 4) in non-negative integers. Furthermore, there are many Diophantine equations in the form $m^x + n^y = z^2$ where $m = 10a + b$, $n = 10b + a$ with a, b are in {0, 1, 2, ..., 9}. The methods to find non-negative solutions of them are still open problems.

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