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บทความวิจัย

มอดูลพัวส์ซองเชิงเดียวที่มีมิติจำกัดของพีชคณิตพัวส์ซองบางรูป โดยใช้วิธีของพีชคณิตลีบนไอดีลใหญ่สุดเฉพาะกลุ่มพัวส์ซอง

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บทคัดย่อ

เราศึกษาพีชคณิตพัวส์ซองในรูป $A = \mathbb{C}[x, y, z]$ ที่มีตัวก่อกำเนิดจำนวนสามตัวและความสัมพันธ์สามแบบซึ่งเป็นความสัมพันธ์แบบเดียวกับความสัมพันธ์ของ T ใน Changtong et al. (2018). เราจำแนกมอดูลพัวส์ซองเชิงเดียวที่มีมิติจำกัดของพีชคณิตพัวส์ซอง A โดยการพิจารณา J/J^2 ที่มีโครงสร้างของพีชคณิตลี เมื่อ J เป็นไอดีลใหญ่ที่สุดของ A และแสดงว่ามีไอดีลใหญ่ที่สุดเพียงห้า ไอดีลเท่านั้นคือ $J_i, i = 1,2,\ldots,5$ สำหรับแต่ละ $d \geq 1$, มีมอดูลพัวส์ซองเชิงเดียวของพีชคณิตพัวส์ซอง A ที่ถูกกำจัดด้วย $J_i, i = 1,2,\ldots,5$ ที่มีมิติ d อยู่เพียงห้าชุดเท่านั้น

คำสำคัญ: พีชคณิตพัวส์ซอง ไอดีลพัวส์ซอง พัวส์ซองไอดีลใหญ่ที่สุด พีชคณิตอนุพัทธ์ มอดูลพัวส์ซอง

Research Article

On finite-dimensional simple Poisson module on a certain Poisson algebra by using the specific Lie Structure method on Poisson maximal ideals

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Abstract

We study a Poisson algebra $\mathbf{A} = \mathbb{C}[\mathbf{x}, \mathbf{y}, \mathbf{z}]$ with three generators and having three relations the same as the relations of $\boldsymbol{T}$ in Changtong et al. (2018). We classify the finite-dimensional simple Poisson modules over $\boldsymbol{A}$ by considering its Lie structure $\boldsymbol{J}/\boldsymbol{J}^{2}$, where $\boldsymbol{J}$ is a Poisson maximal ideal of $\boldsymbol{A}$. We show that there are only five Poisson maximal ideals $\boldsymbol{J}_{\boldsymbol{i}}, \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{5}$. There are five $\boldsymbol{d}$-dimensional simple Poisson module annihilated by $\boldsymbol{J}_{\boldsymbol{i}}, \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{5}$, for each positive integer $\boldsymbol{d} \geq \mathbf{1}$.

Keywords: Poisson algebras, Poisson ideal, Poisson maximal ideal, Derived algebra, Poisson modules

Introduction

A Poisson algebra A is a commutative $\mathbb{C}$-algebra equipped with a Lie bracket $\{-,-\}$ satisfying the Leibniz rule: $\{ab,c\} = a\{b,c\} + \{a,c\}b$ for all $a,b,c \in A$. The bracket $\{-,-\}$ is then called a Poisson bracket. For a Poisson bracket, it is an important binary operation in Hamiltonian mechanics and playing a central role in Hamilton's equation of motion for classical mechanics and mathematics. In a general sense, the Poisson bracket is used to define a Poisson algebra and it also occurs on the theory of Lie algebras.

If T is a $\mathbb{C}$-algebra with a central non-unit non-zero-divisor t such that $A := T/tT$ is commutative then there is a Poisson bracket $\{-,-\}$ on A such that $\{\bar{x},\bar{y}\} = \overline{t^{-1}[x,y]}$ for all $\bar{x},\bar{y} \in A,$ see Changtong et al. (2018).

Changtong and Sasom (2018) considered Poisson A-modules in the aspect of Jordan (2010) and classified the finite-dimensional simple Poisson modules for the Poisson algebra $A = \mathbb{C}[x,y,z]$ with Poisson bracket :

$$\{x,y\} = yx + x + y + z,$$
$$\{y,z\} = zy + x + y + z, \quad \text{and}$$
$$\{z,x\} = xz + x + y + z\,.$$

This gives two Poisson maximal ideals, say $J_i, i = 1,2,$ and we have seen that every finite-dimensional simple Poisson modules over A annihilated by J_1 is 1-dimensional. For each positive integer $d \geq 1,$ there is a unique d-dimensional simple Poisson module over A annihilated by J_2. Chansuriya and Sasom (2021) also classified the finite-dimensional simple Poisson modules for the same Poisson algebra $A = \mathbb{C}[x,y,z]$ by using the method in the PhD thesis of Sasom (2006).

Chen (2019) classified finite-dimensional simple Poisson modules for the Poisson algebra $A = \mathbb{C}[x,y,z]$ with Poisson bracket :

$$\{x,y\} = 2(z - yx), \quad \{y,z\} = 2(x - zy), \quad \{z,x\} = 2(y - xz).$$

There are five Poisson maximal ideals in this Poisson algebra and five d –dimensional simple Poisson modules for each $d \geq 1$ by using the method in the PhD thesis of Sasom (2006).

In this research, we classify the finite-dimensional simple Poisson modules for the Poisson algebra suggested by D. A. Jordan for the Poisson algebra $A = \mathbb{C}[x,y,z]$ with Poisson bracket :

$$\{x,y\} = yx + z - x - y,$$
$$\{y,z\} = zy + x - y - z, \quad \text{and}$$
$$\{z,x\} = xz + y - x - z.$$

There are five Poisson maximal ideals in this Poisson algebra and we have shown in Changtong et al. (2018) that there are five d-dimensional simple Poisson modules for each positive integer $d \geq 1$ by using direct method in the PhD thesis of Sasom (2006) which is quite difficult to classify so we decided to change the method by following the new techniques which appear in Jordan (2010) and Changtong and Sasom (2018).

Preliminaries

This topic contains some of the materials that will be used throughout this work. The main topics are Lie algebras Dixmier (1996), Derived algebras, Low-dimensional Lie algebras modules Erdmann and Wildon (2006) and Humphreys (1972), Poisson algebras and Poisson and Jordan's result shown in Jordan (2010).

Lie algebras

We now present the definition of a Lie algebra which leads to the definition of a Poisson algebra.

Lie algebra arise as a vector space of linear transformations endowed with a new operation which is in general neither commutative nor associative denoted by $[x,y] = xy - yx$ and called the Lie bracket or commutator of x and y.

Definition 1. Let F be a field. A ***Lie algebra*** over F is an F–vector space L, together with a bilinear map, the Lie bracket $[-,-] : L \times L \longrightarrow L$ given by $(x,y) \mapsto [x,y]$ satisfying the following properties:

1) $[x,x] = 0$ for all $x \in L$,
2) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ for all $x,y,z \in L$.

Axiom 2) is called the **Jacobi identity**. A Lie algebra **commutative** if $[x,y] = 0$ for all $x,y \in L$.

Definition 2. Let L be a Lie algebra. A subspace K of L is called a ***Lie subalgebra*** if $[x,y] \in K$ for all $x,y \in K$.

Definition 3. A subspace I of a Lie algebra L is called a **Lie ideal** of L if for all $x \in L$, $y \in I$, $[x,y] \in I$.

Definition 4. Let L and M be Lie algebras and a **Lie algebra homomorphism** is a linear map between to Lie algebras L and M such that it is compatible with the Lie bracket:

$$\theta: L \to M \text{ and } \theta([x,y]) = [\theta(x), \theta(y)] \text{ for all } x,y \in L.$$

Definition 5. A Lie algebra L is called **irreducible** if the only ideals in L are L and $\{0\}$. A Lie algebra L is called **simple** if it is irreducible and $\dim L \geq 2$.

Example 6. The Lie algebra $\mathrm{sl}(2,\mathbb{C})$ is identified with the Lie algebra of 2×2 matrices. We defined the following three elements

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

as a basis. The Lie products among e, f, and h are given by

$$[e,f] = h, \ [h,e] = 2e \text{ and } [h,f] = -2f.$$

It is well known that the Lie algebra $\mathrm{sl}(2,\mathbb{C})$ is 3-dimensional.

Proposition 7. (Hall, 2015) The Lie algebra $\mathrm{sl}(2,\mathbb{C})$ is simple.

Derived algebras

The derived algebra is one of the tools to investigate the results. Let I and J be ideals of a Lie algebra L, we define a product of ideals $[I,J]$ to be the span of the commutators of elements of I and J, that is $[I,J] := \mathrm{Span}\{[x,y]: x \in I, y \in J\}$. When we take $I = J = L$, we obtain $[L,L]$.

Definition 8. The ideal $[L,L]$ is called the ***derived algebra*** of L if it consists of all linear combinations of commutators $[x,y]$ where $x,y \in L$, that is, $[L,L] := \{\sum x_i y_i : x_i \in L, y_i \in L\}$ is an ideal.

Evidently, L is ***abelian*** or ***commutative*** if and only if $[L,L] = 0$. Note that a one-dimensional Lie algebra is necessary commutative since $[ax,bx] = 0$ for all $x \in L$ and all scalars a and b.

Example 9. Let $\mathrm{gl}(\mathrm{n},\mathrm{F})$ be the set of $\mathrm{n} \times \mathrm{n}$ matrices over a field F and $\mathrm{sl}(\mathrm{n},\mathrm{F})$ be the subspace of $\mathrm{gl}(\mathrm{n},\mathrm{F})$ consisting of all matrices of traces 0. Recall that the trace of a square matrix is the sum of its diagonal entries. Then $[\mathrm{gl}(\mathrm{n},\mathrm{F}), \mathrm{gl}(\mathrm{n},\mathrm{F})] = \mathrm{sl}(\mathrm{n},\mathrm{F})$ and $[\mathrm{sl}(\mathrm{n},\mathrm{F}), \mathrm{sl}(\mathrm{n},\mathrm{F})] = \mathrm{sl}(\mathrm{n},\mathrm{F})$.

The following theorem is very useful for classifying the finite-dimensional simple Lie algebras.

Theorem 10. (Erdmann and Wildon, 2006) Suppose that L is a 3-dimensional complex Lie algebra such that $[L,L]=L$. Then $L\cong \mathrm{sl}(2,\mathbb{C})$.

Poisson Algebras and Poisson modules

In this topic, we present definition of Poisson algebras, Poisson modules and a Lie algebra J/J^2, where J is a Poisson maximal ideal.

Definition 11. Let A be a finitely generated commutative algebra over $\mathbb{C}$. A ***Poisson bracket*** on A is a Lie algebra bracket $\{-,-\}$ satisfying the Leibniz rule : $\{\mathrm{ab},\mathrm{c}\}=\mathrm{a}\{\mathrm{b},\mathrm{c}\}+\{\mathrm{a},\mathrm{c}\}\mathrm{b}$ for all $a,b,c\in A$. The pair $(\mathrm{A},\{-,-\})$ is called a ***Poisson algebra***.

Definition 12. A subalgebra B of A is a ***Poisson subalgebra*** of A if $\{b,c\}\in B$ for all $b,c\in B$.

Definition 13. An ideal I of a Poisson algebra A is a ***Poisson ideal*** if $\{i,a\}\in I$ for all $i\in I$ and all $a\in A$. If I is a Poisson ideal of A then A/I is a Poisson algebra in the obvious way:

$$\{a+I,b+I\}=\{a,b\}+I \text{ for all } a,b\in A$$

A Poisson algebra A is said to be ***simple*** if its only Poisson ideals are $\mathbf{0}$ and A.

Definition 14. A Poisson ideal I of a Poisson algebra A is called a **maximal Poisson ideal** if J is a Poisson ideal and $I\nsubseteq J$ then $I=A$. An ideal I of a Poisson algebra A is said to be a **Poisson maximal ideal** if I is a maximal ideal of A and also a Poisson ideal.

Next, we present the definition of Poisson module. There is more than one definition of Poisson module in the literature. We shall use the one introduced by Farkas (2000) and Oh (1999).

Definition 15. Let A be a commutative Poisson algebra with Poisson bracket $\{-,-\}$. An A-module M is a ***Poisson A-module*** if there is a bilinear form $\{-,-\}_M: A\times M\to M$ such that the following axioms hold for all $a,b\in A$ and all $m\in M$:

(i) $\{a,bm\}_M=\{a,b\}\,m+b\{a,m\}_M$;

(ii) $\{ab,m\}_M=a\,\{b,m\}_M+b\{a,m\}_M$;

(iii) $\{\{a,b\},m\}_M=\{a,\{b,m\}_M\}_M-\{b,\{a,m\}_M\}_M$.

A submodule N of a Poisson module M is called a ***Poisson submodule*** if $\{a,n\}_M\in N$ for all $a\in A$ and $n\in N$.

Definition 16. Let N be a left module over a ring R. Give any subset $X\subseteq N$, the ***annihilator*** of X is the set $ann_R(X)=\{r\in R: rx=0 \text{ for all } x\in X\}$, which is a left ideal of R.

Definition 17. Let M be a Poisson module over a Poisson algebra A and let $S\subseteq M$. In the module sense, we denote the annihilator of S in A by

$$ann_A(S)=\{r\in A: rx=0 \text{ for all } x\in S\}$$

and we denote the set

$$Pann_A(S)=\{a\in A: \{a,m\}_M=0 \text{ for all } m\in S\}.$$

Lemma 18. (Jordan, 2010) Let A be a Poisson algebra and M be a Poisson A-module. Let $J = ann_A(M)$. Then

1. J is a Poisson ideal of A.

2. if M is a finite-dimensional simple Poisson module then J is a Poisson maximal ideal of A.

3. $\mathbb{C} + J^2 \subseteq Pann_A M$.

Definition 19. Let L and M be Poisson algebras. An algebra homomorphism θ between Poisson algebras L and M is a **Poisson homomorphism** if it is also a Lie homomorphism.

Definition 20. An algebra homomorphism between two Poisson algebras is a **Poisson isomorphism** if it is a bijective Poisson homomorphism. Two Poisson brackets on the same algebra A are **equivalent** if the corresponding Poisson algebras are isomorphic. Let $(R, \{-,-\})$ be a Poisson algebra. We say that a $\mathbb{C}$-algebra automorphism $\theta: R \to R$ is a **Poisson automorphism** if for all $x, y \in R$, $\theta(\{x,y\}) = \{\theta(x), \theta(y)\}$. Poisson automorphisms can be used to twist the module structures of a Poisson modules, as specified in the following theorem.

Theorem 21. (Sasom, 2006) Let R be a commutative Poisson algebra with a $\mathbb{C}$-algebra automorphism α. Let $r, s, t \in R$. If $\{\alpha(r), \alpha(s)\} = \alpha\{r, s\}$ and $\{\alpha(r), \alpha(t)\} = \alpha\{r, t\}$ then $\{\alpha(r), \alpha(s+t)\} = \alpha\{r, s+t\}$ and $\{\alpha(r), \alpha(st)\} = \alpha\{r, st\}$. Therefore if X is a set of generators of R and $\alpha(\{x,y\}) = \{\alpha(x), \alpha(y)\}$ for all $x, y \in X$ then α is a Poisson automorphism of R.

Definition 22. Let L and M be Poisson A-modules. A **Poisson module homomorphism** $\theta: L \to M$ is an A-module homomorphism such that $\theta(\{a,m\}_L) = \{a, \theta(m)\}_M$ for all $a \in A$ and $m \in M$. If a map θ is bijective then $\theta^{-1}: M \to L$ is also Poisson module homomorphism and θ is a Poisson module isomorphism.

Remarks If $\theta: A \to B$ is a homomorphism of Lie algebras and M is a B-module then there is an A-module M^θ such that $M = M^\theta$, as a set, and $[a,m]_{M^\theta} = [\theta(a), m]_M$ for all $a \in A$ and all $m \in M$.

Lie algebra J/J^2 where J is a Poisson maximal ideal

Let A be Poisson algebra and let I and J be Poisson ideals of A. Then IJ is a Poisson ideal of A. Of course I and J are Lie subalgebra of A under $\{-,-\}$. If $I \subseteq J$, then I is a Lie ideal of J and J/I is a Lie algebra. In particular, J/J^2 is always a Lie algebra and denoted by $\mathfrak{g}(J) := J/J^2$.

Studying Poisson modules, one natural way to find Poisson modules is, for I and J are Poisson ideals of A with $I \subseteq J$, the factor J/I is a Poisson A-module with the Poisson bracket, for $a \in A$ and $j \in J$, $\{a, j+I\}_{J/I} = \{a,j\}_J + I$. We can check that $\{-,-\}_{J/I}$ is well-defined, and all the axioms for a Poisson module are hold. By above argument, J/I is also a Lie algebra. Every Poisson subalgebra of J/I is a Lie ideal, so if J/I is simple as a Lie algebra, then it is simple as a Poisson module. If A is affine and J is a Poisson maximal ideal, so that $A = J + \mathbb{C}$, then the converse is also true because every Lie ideal of J/I is then a Poisson A-submodule. If I and J are Poisson ideals of a Poisson algebra A, then I/IJ and J/JI are Poisson modules. Recall that an affine Poisson algebra is a Poisson algebra which is finitely generated as a $\mathbb{C}$-algebra.

The following is the main result by Jordan (2010). We use this result to tackle the later research problems. Jordan proves the result giving a method to determine the finite-dimensional simple Poisson modules over any affine Poisson algebra as the following Theorem.

Theorem 23. (Jordan, 2010) Let A be an affine generated Poisson algebra.

(i) Let M be a finite-dimensional simple Poisson A-module and let $J = ann_A(M)$. There is a simple module M^* for the Lie algebra $\mathcal{g}(J)$ such that $M^* = M$, as $\mathbb{C}$-vector space, and

$$[j + J^2, m]_{M^*} = \{j, m\}_M \quad \text{for all } j \in J \text{ and } m \in M.$$

(ii) Let J be a Poisson maximal ideal of A and let N be a finite-dimensional simple $\mathcal{g}(J)$-module. There exists a simple Poisson A-module N' and a Lie homomorphism $\theta: A \to \mathcal{g}(J)$ such that $N' = N^\theta$ as a Lie module over A and $J = ann_A(N')$.

(iii) For all finite-dimensional simple Poisson modules M, $M^{*\prime} = M$. For all Poisson maximal ideals J of A and all finite-dimensional simple $\mathcal{g}(J)$-modules N, $N' = N$.

(iv) The procedure in (i) and (ii) establish a bijection Γ from the set of isomorphism classes of finite-dimensional simple Poisson module over A to the set of pairs $(J, \widehat{N})$, where J is a Poisson maximal ideal of A and N is a finite-dimensional simple $\mathcal{g}(J)$-module, given by $\Gamma(\widehat{M}) = (ann_A(M), \widehat{M^*})$ where $\widehat{M}$ is isomorphism class of M and $\widehat{M^*}$ is isomorphism class of M^*

Results

Let T be the $\mathbb{C}$-algebra generated by x, y, z, t and t^{-1} subject to the relations

$$xy - tyx = (t-1)(z - x - y),$$
$$yz - tzy = (t-1)(x - y - z),$$
$$zx - txz = (t-1)(y - x - z), \quad \text{and}$$
$$xt = tx, \;\; yt = ty, \;\; zt = tz, \;\; tt^{-1} = 1 = t^{-1}t.$$

Then t is a central element of T.

Let $A \coloneqq T/(t-1)T \simeq \mathbb{C}[x, y, z]$, which is a commutative polynomial algebra. The induced Poisson bracket on A is such that

$$\begin{aligned}\{x, y\} &= \tfrac{1}{t-1}[x, y] \\ &= \tfrac{1}{t-1}(xy - yx) \\ &= \tfrac{1}{t-1}(tyx + (t-1)(z - x - y) - yx) \\ &= yx + z - x - y.\end{aligned}$$

Here we are abusing notation by writing x, y and z for both an element of T and its image in A. Similarly, we obtain $\{y, z\} = zy + x - y - z$, $\{z, x\} = xz + y - x - z$.

In the next lemma, we find the Poisson maximal ideals of A for this Poisson bracket.

Lemma 24. (Changtong el al., 2018) In the above Poisson algebra A, there are only five Poisson maximal ideals of A which are the followings:

$$\begin{aligned}J_1 &= (x-1)A + (y-1)A + (z-1)A, \\ J_2 &= xA + yA + zA, \\ J_3 &= xA + (y-2)A + (z-2)A, \\ J_4 &= (x-2)A + yA + (z-2)A, \quad \text{and} \\ J_5 &= (x-2)A + (y-2)A + zA.\end{aligned}$$

Theorem 25. For $d \geq 1$, there is a unique d-dimensional simple Poisson module over A annihilated by J_1.

Proof. For $J_1 = (x-1)A + (y-1)A + (z-1)A$, we consider the Lie algebra $\mathcal{g}(J_1)$.

Let $u = x-1, v = y-1$ and $w = z-1$. Then $J_1 = uA + vA + wA$ and therefore u, v, w generate A. Hence, in J_1, the Poisson bracket

$$\{u,v\} = \{x-1, y-1\} = \{x,y\} - \{1,y\} - \{x,1\} + \{-1,-1\} = yx + z - x - y$$

Replacing $x = u+1$, $y = v+1$ and $z = w+1$, we obtain $\{u,v\} = vu + w$.. Similarly, $\{v,w\} = wv + u$ $\{w,u\} = uw + v$.

The Lie algebra $\mathcal{g}(J_1) = J_1/J_1^2$ has a basis (the images of) u, v, w and the Lie brackets

$$[u,v] = \{u + J_1^2, v + J_1^2\} = \{u,v\} + J_1^2 = vu + w + J_1^2$$

but $vu \in J_1^2$. Therefore $u,v] = w$. Similarly, we have $[v,w] = u$ and $[w,u] = v$.

We will show that $[u,v], [v,w], [w,u]$ are linearly independent. Let a, b, c be such that $a[u,v] + b[v,w] + c[w,u] = 0$. Then $aw + bu + cv = 0$ and since u, v, w are linearly independent, we obtain $a = b = c = 0$. Therefore $[u,v], [v,w], [w,u]$ are linearly independent. Hence $\mathcal{g}(\mathrm{J}_1)$ is 3-dimensional Lie algebra and the linear map $\theta: sl_2(\mathbb{C}) \to \mathcal{g}(\mathrm{J}_1)$ define on a basis elements by $\theta(e) = w$, $\theta(f) = u$ and $\theta(h) = v$. This implies $\mathcal{g}(\mathrm{J}_1)$ is isomorphic to $sl_2(\mathbb{C})$ by Theorem 10 and it follows from Theorem 23 that, for each $d \geq 1$, A has a unique dimensional simple Poisson module annihilated by J_1.

Theorem 26. For $d \geq 1$, there is a unique d-dimensional simple Poisson module over A annihilated by J_2.

Proof. For $J_2 = xA + yA + zA$, we consider the Lie algebra $\mathcal{g}(J_2)$. Hence, in J_2, the Poisson brackets are

$$\{\mathrm{x},\mathrm{y}\} = \mathrm{yx} + \mathrm{z} - \mathrm{x} - \mathrm{y},$$

$$\{y,z\} = zy + x - y - z,$$

$$\{z,x\} = xz + y - x - z.$$

The Lie algebra $\mathcal{g}(J_2) = J_2/J_2^2$ has a basis (the images of) x, y, z and the Lie bracket

$$[\mathrm{x},\mathrm{y}] = \{\mathrm{x} + J_2^2,\ \mathrm{y} + J_2^2\} = \{\mathrm{x},\mathrm{y}\} + J_2^2 = \mathrm{yx} + \mathrm{z} - \mathrm{x} - \mathrm{y} + J_2^2,$$

But $yx \in J_2^2$. Therefore $[\mathrm{x},\mathrm{y}] = z - x - y$. Similarly, $[\mathrm{y},\mathrm{z}] = x - y - z$ and $[z,x] = y - x - z$.

We will show that $[x,y]$, $[y,z]$, $[z,x]$ are linearly independent. Let a, b, c be such that $a[x,y] + b[y,z] + c[z,x] = 0$. Then $a(z-x-y) + b(x-y-z) + c(y-x-z) = 0$ and $(-a+b-c)x + (-a-b+c)y + (a-b-c)z = 0.$ Since x, y, z are linearly independent and by solving these equations,

$$(-a+b-c) = 0, \quad (-a-b+c) = 0, \quad \text{and} \quad (a-b-c) = 0$$

We obtain $a = b = c = 0$ and so $[x,y]$, $[y,z]$, $[z,x]$ are linearly independent. Hence $\mathcal{g}(J_2)$ is 3-dimensional Lie algebra. This implies that the derived algebra $[\mathcal{g}(J_2), \mathcal{g}(J_2)] = \mathcal{g}(J_2)$. Therefore $\mathcal{g}(J_2)$ is isomorphic to $sl_2(\mathbb{C})$ by Theorem 10 and it follows from Theorem 23 that, for each $d \geq 1$, A has a unique dimensional simple Poisson module annihilated by J_2.

Theorem 27. For $d \geq 1$, there is a unique d-dimensional simple Poisson module over A annihilated by J_3.

Proof. For $J_3 = xA + (y-2)A + (z-2)A$, we consider the Lie algebra $\mathcal{g}(J_3)$.

Let $u = x$, $v = y - 2$ and $w = z - 2$. Then $J_3 = uA + vA + wA$ and therefore u, v, w generate A. Hence, in J_3, the Poisson brackets

$$\{u, v\} = \{x, y - 2\} = \{x, y\} + \{x, -2\} = \{x, y\} = yx + z - x - y.$$

Replacing $x = u, y = v + 2$ and $z = w + 2$, we obtain $\{u, v\} = vu + 2u + w + 2 - u - v - 2 = vu + u + w - v$. Similarly, the Poisson bracket $\{v, w\} = wv + u + w + v$, and $\{w, u\} = uw + u + v - w$.

The Lie algebra $\mathcal{g}(J_3) = J_3/J_3^2$ has a basis (the images of) u, v, w and the Lie brackets

$$[u, v] = \{u + J_3^2, v + J_3^2\} = \{u, v\} + J_3^2 = vu + u + w - v + J_3^2$$

but $vu \in J_3^2$. Therefore $[u, v] = u + w - v$. Similarly, $[v, w] = u + w + v$ and $[w, u] = u + v - w$.

We will show that $[u, v]$, $[v, w]$, $[w, u]$ are linearly independent. Let a, b, c be such that $a[u, v] + b[v, w] + c[w, u] = 0$. Then $a(u + w - v) + b(u + w + v) + c(u + v - w) = 0$ and $(a + b + c)u + (a + b - c)w + (-a + b + c)v = 0$. Since u, w, v are linearly independent and by solving these equations,

$$a + b + c = 0, \quad a + b - c = 0, \quad \text{and} \quad -a + b + c = 0.$$

We obtain $a = b = c = 0$. Therefore $[u, v]$, $[v, w]$, $[w, u]$ are linearly independent. We see that $\mathcal{g}(J_3)$ has 3-dimensional Lie algebra. This implies that the derived algebra $[\mathcal{g}(J_3), \mathcal{g}(J_3)] = \mathcal{g}(J_3)$. Therefore $\mathcal{g}(J_3)$ is isomorphic to $sl_2(\mathbb{C})$ by Theorem 10 and it follows from Theorem 23 that, for each $d \geq 1$, A has a unique dimensional simple Poisson module annihilated by J_3. By using the Poisson module isomorphism, we define $\theta(J_3) = J_4$, where $\theta(x) = y, \theta(y) = x$ and $\theta(z) = z$ and $\theta(J_3) = J_5$ where $\theta(x) = z, \theta(y) = y$ and $\theta(z) = x$, the same conclusion holds for J_4 and J_5, for each of which the d- dimensional simple Poisson module annihilated by J_4 and J_5.

Conclusion and Discussion

The classifying finite-dimensional simple Poisson modules with the method as in Changtong et al. (2018) is quite complicated so we try to work out in the same way as Jordan (2010) which is much easier to get the same result as in Changtong et al. (2018). Hence there are five d-dimensional simple Poisson module for each positive integer $d \geq 1$.

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