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Integral representations of the Pell and Pell-Lucas numbers

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Abstract

In this paper, integral representations of the Pell number P_{kn+r} and the Pell-Lucas number Q_{kn+r} are presented. Using Binet's formula for the Pell and Pell-Lucas numbers establishes some identities equipped with using simple integral calculus prove the integral representations.

Keywords: Pell number, Pell-Lucas number, Integral representation

บทความวิจัยทางวิทยาศาสตร์

ตัวแทนอินทิกรัลของจำนวนเพลล์และเพลล์-ลูคัส

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บทคัดย่อ

ในบความนี้ ได้นำเสนอตัวแทนอินทิกรัลของจำนวนเพลล์ P_{kn+r} และเพลล์-ลูคัส Q_{kn+r} โดยใช้สูตรไบเนตสำหรับจำนวนเพลล์และเพลล์-ลูคัสในการสร้างเอกลักษณ์บางประการและใช้แคลคูลัสอินทิกรัลพื้นฐานในการพิสูจน์

คำสำคัญ: จำนวนเพลล์ จำนวนเพลล์-ลูคัส ตัวแทนอินทิกรัล

Introduction

Several integer sequences are all practical branches of modern science. For instance, the Fibonacci and the Lucas sequence are a well-known mathematical series that has been extensively researched in the literatures. The Fibonacci sequence has been generalized in a variety of ways, including preserving the initial conditions and the recurrence relation. Recall that the nth Fibonacci number F_n is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, for all $n \ge 2$ with $F_0 = 0$ and $F_1 = 1.$ It is well known that the representations of the Fibonacci numbers are given by the following two identities:

Binet's formula (Koshy, 2018) for the Fibonacci number is

$$F_n = \frac{1}{\sqrt{5}}\left(\alpha^n - \frac{(-1)^n}{\alpha^n}\right), \text{ where } \alpha = \frac{1+\sqrt{5}}{2}, \text{ and}$$

Simpson formula or *Cassini identity* for the Fibonacci number is

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n.$$

The nth Lucas number L_n is defined by the recurrence relation $L_n = L_{n-1} + L_{n-2}$, for $n \ge 2$ with $L_0 = 2$ and $L_1 = 1.$

Binet's formula (Koshy, 2018) for the Lucas number is

$$L_n = \alpha^n + \frac{(-1)^n}{\alpha^n}, \text{ where } \alpha = \frac{1+\sqrt{5}}{2}.$$

Simpson formula for the Lucas number is

$$L_{n-1}L_{n+1} - L_n^2 = 5(-1)^{n+1}.$$

In 2015 (Glasser and Zhou, 2015) Glasser and Zhou have given an integral representation for the Fibonacci number:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1^n}{2}\right) - \frac{2}{\pi}\int_0^\infty \left(\frac{\sin(x/2)}{x}\right)\left(\frac{\cos(nx) - 2\sin(nx)\sin x}{5\sin^2 x + \cos^2 x}\right)dx.$$

Moreover, integral representations for the even and odd terms in the Fibonacci number are

$$F_{2n} = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^{2n} - \frac{2}{\pi}\int_0^\infty \left(\frac{\sin(x/2)}{x}\right)\left(\frac{\cos(2nx)}{5\sin^2 x + \cos^2 x}\right)dx, \text{ and}$$

$$F_{2n+1} = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^{2n+1} + \frac{4}{\pi}\int_0^\infty \left(\frac{\sin(x/2)}{x}\right)\left(\frac{\sin\left[(2n+1)x\right]\sin x}{5\sin^2 x + \cos^2 x}\right)dx,$$

where n is a non-negative integer.

In 2022 (Stewart, 2022) Stewart has used the connection between the Fibonacci numbers, the Lucas numbers, the golden ratio $\alpha^n = \frac{L_n + \sqrt{5}F_n}{2}$, and identity $L_n^2 - 5F_n^2 = 4(-1)^n$ to give the integral representations of the Fibonacci number F_{kn} and Lucas number L_{kn}:

$$F_{kn} = \frac{nF_k}{2^n}\int_{-1}^{1}\left(L_k + \sqrt{5}F_k x\right)^{n-1}dx,$$
$$L_{kn} = \frac{1}{2^n}\int_{-1}^{1}\left(L_k + (n+1)\sqrt{5}F_k x\right)\left(L_k + \sqrt{5}F_k x\right)^{n-1}dx,$$

where n is a non-negative integer and k is a positive integer. The special cases of such identities for $k=1$ are also discussed in Stewart's paper (Stewart, 2023). Moreover, other integral representations for the Fibonacci number F_{kn+r} and Lucas number L_{kn+r}:

$$F_{kn+r} = \frac{1}{2^{n+1}}\int_{-1}^{1}\left(nF_kL_r + F_rL_k + (n+1)\sqrt{5}F_kF_r x\right)\left(L_k + \sqrt{5}F_k x\right)^{n-1}dx,$$
$$L_{kn+r} = \frac{1}{2^{n+1}}\int_{-1}^{1}\left(5nF_kF_r + L_kL_r + (n+1)\sqrt{5}F_rL_r x\right)\left(L_k + \sqrt{5}F_k x\right)^{n-1}dx,$$

where n, r are non-negative integers and k is a positive integer (Stewart, 2022).

Like Fibonacci and Lucas numbers, the Pell family is widely used. Pell and Pell–Lucas numbers also provide boundless opportunities to experiment, explore, and conjecture. Further details can be found, for instance, (Trojnar-Spelina and Włoch, 2019; Erduvan and Keskin, 2022). The nth Pell number P_n and the nth Pell-Lucas number Q_n are explicitly given by the Binet-type formulas: $P_n = \frac{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n}{2\sqrt{2}}$ and $Q_n = \left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n$, respectively.

In this paper, we give new integral representations that have never existed before of the Pell number and the Pell-Lucas number by using Binet's formula for the Pell and Pell-Lucas numbers to establish some identities and simple integral calculus to prove them.

Preliminaries

In this section, we briefly recall some of the concepts and results that we will require.

The nth Pell number P_n is defined by the recurrence relation $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$ with $P_0 = 0$ and $P_1 = 1$. The first few terms of the sequence are $0, 1, 2, 5, 12, 29, 70, 169, 408, 985, \ldots$ *Binet's formula* (Bicknell, 1915) for the Pell number is

$$P_n = \frac{1}{2\sqrt{2}}\left(\varphi^n - \frac{(-1)^n}{\varphi^n}\right), \tag{1.1}$$

where $\varphi = 1 + \sqrt{2}$.

Simpson formula (Horadam, 1971) for the Pell number is

$$P_{n-1}P_{n+1}-P_n^2=(-1)^n.$$

The nth Pell-Lucas number Q_n is defined by the recurrence relation $Q_n=2Q_{n-1}+Q_{n-2}$, for $n\geq 2$ with $Q_0=2$ and $Q_1=2$. The first few terms of the sequence are $2, 2, 6, 14, 34, 82, 198, 478,\ldots$ *Binet's formula* (Bicknell, 1915) for the Pell-Lucas number is

$$Q_n=\varphi^n+\frac{(-1)^n}{\varphi^n}, \tag{1.2}$$

where $\varphi=1+\sqrt{2}$.

Simpson formula (Horadam and Mahon, 1985) for the Pell-Lucas number is

$$Q_{n-1}Q_{n+1}-Q_n^2=8(-1)^{n+1}.$$

Lemma 2.1 Let n be a non-negative integer. Then the following hold:

1. $\varphi^n=\dfrac{Q_n+\sqrt{8}P_n}{2}$,
2. $Q_n^2-8P_n^2=4(-1)^n$,
3. $\dfrac{1}{\varphi^n}=\dfrac{(-1)^n}{2}\left(Q_n-\sqrt{8}P_n\right)$.

Proof. 1. Combining Binet's formula for the Pell (1.1) and Pell-Lucas numbers (1.2), gives

$$Q_n+2\sqrt{2}P_n=\left(\varphi^n+\frac{(-1)^n}{\varphi^n}\right)+\left(\varphi^n-\frac{(-1)^n}{\varphi^n}\right)=2\varphi^n.$$

Then $\varphi^n=\dfrac{Q_n+\sqrt{8}P_n}{2}$ which completes the proof.

2. Squaring both sides of Binet's formula for the Pell (1.1) and Pell-Lucas numbers (1.2) and subtracting them, give

$$Q_n^2-8P_n^2=\left(\varphi^{2n}+2(-1)^n+\left(\frac{(-1)}{\varphi}\right)^{2n}\right)-\left(\varphi^{2n}-2(-1)^n+\left(\frac{(-1)}{\varphi}\right)^{2n}\right)=4(-1)^n,$$

which completes the proof.

3. It follows from (1) that

$$\frac{1}{\varphi^n}=\frac{2}{Q_n+\sqrt{8}P_n}=\frac{2\left(Q_n-\sqrt{8}P_n\right)}{Q_n^{\ 2}-8P_n^{\ 2}}=\frac{2\left(Q_n-\sqrt{8}P_n\right)}{4(-1)^n}=\frac{(-1)^n}{2}\left(Q_n-\sqrt{8}P_n\right).$$

Lemma 2.2 Let m and r be non-negative integers. Then the following hold:

1. $2P_{m+r}=P_mQ_r+P_rQ_m$,
2. $2Q_{m+r}=Q_mQ_r+8P_mP_r$.

Proof. Using Binet's formula for the Pell (1.1) and Pell-Lucas numbers (1.2), we get

$$\begin{aligned} P_mQ_r + P_rQ_m &= \frac{1}{\sqrt{8}}\left(\varphi^m - \frac{(-1)^m}{\varphi^m}\right)\left(\varphi^r + \frac{(-1)^r}{\varphi^r}\right) + \frac{1}{\sqrt{8}}\left(\varphi^r - \frac{(-1)^r}{\varphi^r}\right)\left(\varphi^m + \frac{(-1)^m}{\varphi^m}\right) \\ &= \frac{2}{\sqrt{8}}\left(\varphi^{m+r} - \frac{(-1)^{m+r}}{\varphi^{m+r}}\right) \\ &= 2P_{m+r}, \end{aligned}$$

and $$\begin{aligned} Q_mQ_r + 8P_mP_r &= \left(\varphi^m + \frac{(-1)^m}{\varphi^m}\right)\left(\varphi^r + \frac{(-1)^r}{\varphi^r}\right) + 8\cdot\frac{1}{\sqrt{8}}\left(\varphi^m - \frac{(-1)^m}{\varphi^m}\right)\cdot\frac{1}{\sqrt{8}}\left(\varphi^r - \frac{(-1)^r}{\varphi^r}\right) \\ &= 2\left(\varphi^{m+r} + \frac{(-1)^{m+r}}{\varphi^{m+r}}\right) = 2Q_{m+r}. \end{aligned}$$

Main Results

In the first of this section, the integral representation for the Pell number P_{kn} can be found by employing other known relations between the two numbers P_k and Q_k.

Theorem 3.1 For n is a non-negative integer and k is a positive integer, the Pell number P_{kn} can be represented by the integral

$$P_{kn} = \frac{nP_k}{2^n}\int_{-1}^{1}\left(Q_k + \sqrt{8}P_kx\right)^{n-1}dx. \tag{3.1}$$

Proof. Straightforward simple integration leads to

$$\frac{nP_k}{2^n}\int_{-1}^{1}\left(Q_k + \sqrt{8}P_kx\right)^{n-1}dx = \frac{1}{\sqrt{8}}\left[\left(\frac{Q_k + \sqrt{8}P_kx}{2}\right)^n\right]_{-1}^{1} = \frac{1}{\sqrt{8}}\left[\left(\frac{Q_k + \sqrt{8}P_k}{2}\right)^n - \left(\frac{Q_k - \sqrt{8}P_k}{2}\right)^n\right]. \tag{3.2}$$

Applying Lemma 2.1 (1) and (3) in (3.2) with n replaced with k, and replace k by kn yields

$$\frac{nP_k}{2^n}\int_{-1}^{1}\left(Q_k + \sqrt{8}P_kx\right)^{n-1}dx = \frac{1}{\sqrt{8}}\left(\varphi^k - \frac{(-1)^k}{\varphi^k}\right) = \frac{1}{\sqrt{8}}\left(\varphi^{kn} - \frac{(-1)^{kn}}{\varphi^{kn}}\right) = P_{kn}.$$

This completes the proof.

The integral representations of the Pell number for even and odd orders are shown as follow:

Corollary 3.2 Let n be a non-negative integer.

1. The Pell number P_{2n} can be represented by the integral

$$P_{2n} = n\int_{-1}^{1}\left(3 + \sqrt{8}x\right)^{n-1}dx. \tag{3.3}$$

2. The Pell number P_{2n+1} can be represented by the integral

$$P_{2n+1}=\frac{1}{2}\int_{-1}^{1}\left(2n+3+(n+1)\sqrt{8}x\right)\left(3+\sqrt{8}x\right)^{n-1}dx. \tag{3.4}$$

Proof. 1. Notice that $P_2=2$ and $Q_2=6.$ Setting $k=2$ in (3.1) gives

$$\begin{aligned}P_{2n}&=\frac{nP_2}{2^n}\int_{-1}^{1}\left(Q_2+\sqrt{8}P_2x\right)^{n-1}dx\\&=\frac{n}{2^{n-1}}\int_{-1}^{1}\left(6+2\sqrt{8}x\right)^{n-1}dx\\&=n\int_{-1}^{1}\left(3+\sqrt{8}x\right)^{n-1}dx.\end{aligned}$$

Thus, (3.3) holds.

2. Reindexing of n by $n+1$ in (3.3), we get

$$P_{2n+2}=(n+1)\int_{-1}^{1}\left(3+\sqrt{8}x\right)^{n}dx. \tag{3.5}$$

Using $P_{2n+2}=2P_{2n+1}+P_{2n}$ with (3.3) and (3.5), we obtain

$$\begin{aligned}P_{2n+1}&=\frac{1}{2}\left(P_{2n+2}-P_{2n}\right)\\&=\frac{1}{2}\left((n+1)\int_{-1}^{1}\left(3+\sqrt{8}x\right)^{n}dx-n\int_{-1}^{1}\left(3+\sqrt{8}x\right)^{n-1}dx\right)\\&=\frac{1}{2}\int_{-1}^{1}\left(2n+3+(n+1)\sqrt{8}x\right)\left(3+\sqrt{8}x\right)^{n-1}dx.\end{aligned}$$

This completes the proof.

Remark 3.3 As in Corollary 3.2 (1), P_{2n} is product of n and the region under a curve $y_n(x)=\left(3+\sqrt{8}x\right)^{n-1}$ between -1 and 1. Indeed, (3.3) becomes

$$P_{2n}=n\int_{-1}^{1}y_n\,dx.$$

For example, see Figure 1.

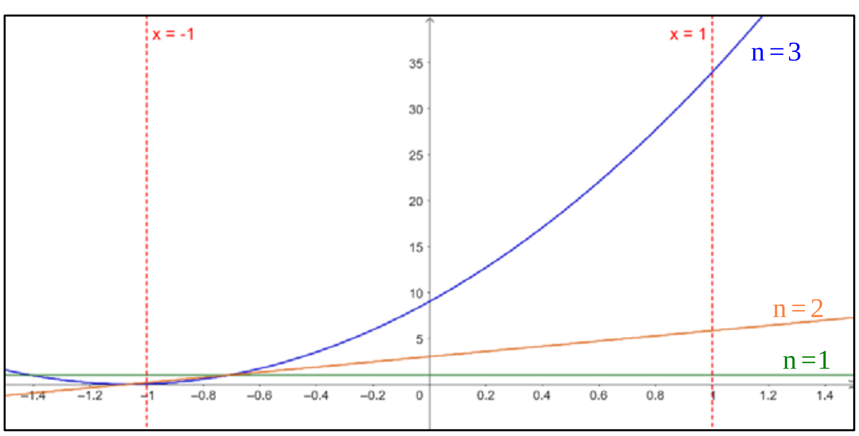

Figure 1 The region under a curve of $y_n(x)$ with $-1\le x\le 1$ for $n=1,2,3.$

Next, we provide the integral representations for the Pell-Lucas number Q_{kn} based on the two numbers P_k and Q_k.

Theorem 3.4 For n is a non-negative integer and k is a positive integer, the Pell-Lucas number Q_{kn} can be represented by the integral

$$Q_{kn} = \frac{1}{2^n}\int_{-1}^{1}\left(Q_k + (n+1)\sqrt{8}P_k x\right)\left(Q_k + \sqrt{8}P_k x\right)^{n-1} dx. \tag{3.6}$$

Proof. Replacing n by $n+1$ in (3.1) becomes

$$P_{kn+k} = \frac{(n+1)P_k}{2^{n+1}}\int_{-1}^{1}\left(Q_k + \sqrt{8}P_k x\right)^n dx. \tag{3.7}$$

Integrating by part (3.6) and using (3.7), we have

$$\frac{1}{2^n}\int_{-1}^{1}\left(Q_k + (n+1)\sqrt{8}P_k x\right)\left(Q_k + \sqrt{8}P_k x\right)^{n-1} dx$$

$$= \frac{1}{n\sqrt{8}P_k}\left[\left(\frac{Q_k + \sqrt{8}P_k x}{2}\right)^n \left(Q_k + (n+1)\sqrt{8}P_k x\right)\right]_{-1}^{1} - \frac{n+1}{n2^n}\int_{-1}^{1}\left(Q_k + \sqrt{8}P_k x\right)^n dx$$

$$= \frac{1}{n\sqrt{8}P_k}\left(\frac{Q_k + \sqrt{8}P_k}{2}\right)^n \left(Q_k + (n+1)\sqrt{8}P_k\right) - \frac{1}{n\sqrt{8}P_k}\left(\frac{Q_k - \sqrt{8}P_k}{2}\right)^n \left(Q_k - (n+1)\sqrt{8}P_k\right) - \frac{2P_{kn+k}}{nP_k}.$$

Applying Lemma 2.1 (1) and (2) to the righthand side of the above equation gives

$$\frac{1}{2^n}\int_{-1}^{1}\left(Q_k + (n+1)\sqrt{8}P_k x\right)\left(Q_k + \sqrt{8}P_k x\right)^{n-1} dx$$

$$= \frac{1}{n\sqrt{8}P_k}\varphi^{kn}\left(Q_k + (n+1)\sqrt{8}P_k\right) - \frac{1}{n\sqrt{8}P_k}\frac{(-1)^{kn}}{\varphi^{kn}}\left(Q_k - (n+1)\sqrt{8}P_k\right) - \frac{2P_{kn+k}}{nP_k}$$

$$= \frac{1}{nP_k}\left[\frac{1}{\sqrt{8}}\varphi^{kn}Q_k + (n+1)\varphi^{kn}P_k - \frac{1}{\sqrt{8}}\frac{(-1)^{kn}}{\varphi^{kn}}Q_k + (n+1)\frac{(-1)^{kn}}{\varphi^{kn}}P_k\right] - \frac{2P_{kn+k}}{nP_k}$$

$$= \frac{1}{nP_k}\left[\frac{1}{\sqrt{8}}\left(\varphi^{kn} - \frac{(-1)^{kn}}{\varphi^{kn}}\right)Q_k + (n+1)\left(\varphi^{kn} + \frac{(-1)^{kn}}{\varphi^{kn}}\right)P_k\right] - \frac{2P_{kn+k}}{nP_k}.$$

Applying both Binet's formulas (1.1) and (1.2) with n replaced by kn and Lemma 2.2 (1) leads to

$$\frac{1}{2^n}\int_{-1}^{1}\left(Q_k + (n+1)\sqrt{8}xP_k\right)\left(Q_k + \sqrt{8}P_k x\right)^{n-1} dx = \frac{1}{nP_k}\left[P_{kn}Q_k + (n+1)P_k Q_{kn}\right] - \frac{2P_{kn+k}}{nP_k}$$

$$= \frac{1}{nP_k}\left[\left(P_{kn}Q_k + P_k Q_{kn}\right) + nP_k Q_{kn}\right] - \frac{2P_{kn+k}}{nP_k}$$

$$= \frac{1}{nP_k}\left[2P_{kn+k} + nP_k Q_{kn}\right] - \frac{2P_{kn+k}}{nP_k}$$

$$= Q_{kn},$$

which completes the proof.

Now, both P_{kn} and Q_{kn} are then used to establish integral representations for the Pell number P_{kn+r} and Pell-Lucas number Q_{kn+r} as the following theorems.

Theorem 3.5 For n and r are non-negative integers and k is a positive integer, the Pell number P_{kn+r} can be represented by the integral

$$P_{kn+r} = \frac{1}{2^{n+1}} \int_{-1}^{1} \left(nP_k Q_r + P_r Q_k + (n+1)\sqrt{8} P_k P_r x \right) \left(Q_k + \sqrt{8} P_k x \right)^{n-1} dx. \tag{3.8}$$

Proof. Using Lemma 2.2 (1) with m replaced by $kn,$ we get

$$P_{kn+r} = \frac{1}{2} \left(P_{kn} Q_r + P_r Q_{kn} \right). \tag{3.9}$$

Applying Theorem 3.1 and Theorem 3.4 leads to

$$\begin{aligned} P_{kn+r} &= \frac{1}{2} \left(\left(\frac{nP_k}{2^n} \int_{-1}^{1} \left(Q_k + \sqrt{8} P_k x \right)^{n-1} dx \right) Q_r + P_r \left(\frac{1}{2^n} \int_{-1}^{1} \left(Q_k + (n+1)\sqrt{8} P_k x \right) \left(Q_k + \sqrt{8} P_k x \right)^{n-1} dx \right) \right) \\ &= \frac{1}{2^{n+1}} \int_{-1}^{1} \left(nP_k Q_r + P_r Q_k + (n+1)\sqrt{8} P_k P_r x \right) \left(Q_k + \sqrt{8} P_k x \right)^{n-1} dx, \end{aligned}$$

which completes the proof.

Theorem 3.6 For n and r are non-negative integers and k is a positive integer, the Pell-Lucas number Q_{kn+r} can be represented by the integral

$$Q_{kn+r} = \frac{1}{2^{n+1}} \int_{-1}^{1} \left(8nP_k P_r + Q_k Q_r + (n+1)\sqrt{8} P_k Q_r x \right) \left(Q_k + \sqrt{8} P_k x \right)^{n-1} dx \tag{3.10}$$

Proof. Using Lemma 2.2 (2) with m replaced by $kn,$ we get

$$Q_{kn+r} = \frac{1}{2} \left(Q_{kn} Q_r + 8P_{kn} P_r \right). \tag{3.11}$$

Applying Theorem 3.1 and Theorem 3.4 leads to

$$\begin{aligned} Q_{kn+r} &= \frac{1}{2} \left(\left(\frac{1}{2^n} \int_{-1}^{1} \left(Q_k + (n+1)\sqrt{8} P_k x \right) \left(Q_k + \sqrt{8} P_k x \right)^{n-1} dx \right) Q_r + 8 \left(\frac{nP_k}{2^n} \int_{-1}^{1} \left(Q_k + \sqrt{8} P_k x \right)^{n-1} dx \right) P_r \right) \\ &= \frac{1}{2^{n+1}} \int_{-1}^{1} \left(8nP_k P_r + Q_k Q_r + (n+1)\sqrt{8} P_k Q_r x \right) \left(Q_k + \sqrt{8} P_k x \right)^{n-1} dx, \end{aligned}$$

which completes the proof.

Conclusion

In this paper, we give integral representations of the Pell numbers and the Pell-Lucas numbers by using Binet's formula for the Pell and Pell-Lucas numbers. We establish some identities and simple integral calculus to prove them. Nevertheless, to find integral representations of several integer sequences, such as the Bell numbers, the Padovan numbers, $k-$Fibonacci numbers, and the $k-$Pell numbers, are still open problems.

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