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Solutions of a Certain Forms of Systems of PDEs and Representations of A_2

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Abstract

This paper is concerned with applications of the representations of A_2 to solutions of certain forms of systems of partial differential equations. This is achieved by using representations of A_2 and intertwining operators. Solutions of the systems of partial differential equations can be found by applying products of the related operators to 1.

Keywords: Representations of A_2; Lie group of class A_2; Lie algebra of class A_2

บทความวิจัยทางวิทยาศาสตร์

ผลเฉลยของระบบสมการเชิงอนุพันธ์ย่อยรูปแบบแน่นอนและตัวแทนของ A_2

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บทคัดย่อ

บทความวิจัยนี้เกี่ยวกับการประยุกต์ของตัวแทนของ A_2 ในการหาผลเฉลยของระบบสมการเชิงอนุพันธ์ย่อยรูปแบบแน่นอน ซึ่งสามารถทำได้โดยการใช้ตัวแทนของ A_2 และตัวดำเนินการอินเตอร์ทไวนิง ผลเฉลยของระบบสมการเชิงอนุพันธ์ย่อยสามารถหาได้จากผลคูณของตัวดำเนินการที่สัมพันธ์กันกับ 1

คำสำคัญ: ตัวแทนของ A_2; ลีกรุ๊ปของคลาส A_2; พีชคณิตลีของคลาส A_2

Introduction

Various methods are considered for the study of differential equations. One of the methods for studying of differential equations is group analysis (Ovsiannikov, 1978). Many applications of group analysis to partial differential equations are collected in the Handbook of Lie Group Analysis of Differential Equations (Ibragimov, 1996). Saenkarun, Loutsiouk and Chunrungsikul (2009) studied solutions of some systems of PDEs through representations of group G_2. In this paper, we study solutions of systems of PDEs in certain forms through representations of group A_2 which the systems of PDEs are more general form than the system was studied by Saenkarun (2009). Consider the system of two partial differential equations as follows:

$$\left[\frac{1}{h'(x)}\frac{\partial}{\partial x}\right]^{s+1}\varphi = 0 \qquad (1)$$

$$\left[\frac{\partial}{\partial y} + h(x)\frac{\partial}{\partial z}\right]^{t+1}\varphi = 0$$

where $h(x)$ is an odd degree polynomial and s, t are non-negative integers. We will find all solutions of the system by examining the Lie algebra of differential operators generated by the linear differential operators

$$A = \frac{1}{h'(x)}\frac{\partial}{\partial x},$$

$$B = \frac{\partial}{\partial y} + h(x)\frac{\partial}{\partial z}.$$

The system can be written as

$$A^{s+1}\varphi = 0,$$

$$B^{t+1}\varphi = 0.$$

The Lie algebra $\mathbb{Z}$ of differential operators generated by A, B and $C = [A, B] = AB - BA$ is the Lie algebra of a three-dimensional nilpotent Lie algebra, which turns out to be isomorphic to a maximal nilpotent subalgebra of the simple Lie algebra of class A_2. This property of the differential operators A and B is useful for studying the solutions of the system of PDEs (1) through an examination of the representations of the Lie group A_2. It turns out that the operators A^{s+1} and B^{t+1} are intertwining operators for some pairs of representations of the group A_2, and so the space of solutions of the system of PDEs (1) has the structure of an irreducible finite-dimensional representation of the group A_2.

Research Objectives

1. To study Lie group and Lie algebra of class A_2.
2. To study representations of Lie algebra of class A_2.
3. To apply representations of Lie algebra A_2 to solutions of a certain forms of systems of PDEs.

Matrix generators for the Lie algebra A_2

We consider the 3×3 matrix generators for Lie algebra A_2 as follows:

$$e_\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, e_{\alpha+\beta} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_{-\alpha} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_{-\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, e_{-(\alpha+\beta)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

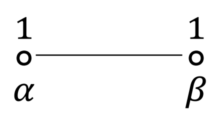
$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, h_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We shall denote by $\mathfrak{g}$ the Lie algebra of class A_2 spanned by h_1, h_2 and the root vectors e_α, e_β, $e_{\alpha+\beta}$, $e_{-\alpha}$, $e_{-\beta}$, $e_{-(\alpha+\beta)}$, which correspond to the roots $\alpha, \beta, \alpha+\beta, -\alpha, -\beta, -(\alpha+\beta)$, where that positive roots are $\alpha, \beta, \alpha+\beta$ and negative roots are $-\alpha, -\beta, -(\alpha+\beta)$ and a simple system of roots (α, β). Let Δ be the set of all roots and Δ^+ be the set of all positive roots. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ spanned by h_1, h_2.

The Cartan matrix is

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

and the corresponding Dynkin diagram is



Constructing representations of A_2 Let G be the Lie group with Lie algebra $\mathfrak{g}$. Then $H_i(t_i) = e^{t_i h_i}$, where $i \in \{1,2\}$ and $E_i(t_i) = e^{t_i e_i}$, where $i \in \Delta$, are one parameter subgroups of Lie group G. Using these one-parameter subgroups of group G, we shall now construct some of its subgroups that will be utilized for constructing some representations of G. The first of these subgroups is a maximal nilpotent subgroup of group G and is constructed as follows:

$$Z_+ = \left\{ \prod_{i \in \Delta^+} E_i(t_i) | t_i \in \mathbb{R} \right\}.$$

For our purpose, we introduce new parameters and we get

$$Z_+ = \begin{pmatrix} 0 & h(x) & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix},$$

where $h(x)$ is an odd degree polynomial.

We also construct another maximal nilpotent subgroup as follows:

$$Z_- = \left\{ \prod_{i \in \triangle^+} E_{-i}(t_{-i}) | t_{-i} \in \mathbb{R} \right\}.$$

The subgroup denoted by H, which is a maximal abelian subgroup of G, is defined as follows:

$$H = \{H_1(t_1)H_2(t_2) | t_1, t_2 \in \mathbb{R}\},$$

and the subgroup denoted by B_-, which is a maximal solvable subgroup of G, is defined as

$$B_- = Z_-H = \{z_-h | z_- \in Z_-, h \in H\}.$$

For $(p, q) \in \mathbb{C}^2$, we define a mapping $\alpha_{p,q}: H \to \mathbb{C}$ by

$$H_1(t_1)H_2(t_2) \mapsto e^{pt_1}e^{qt_2}.$$

By the basic property of the exponential function, $\alpha_{p,q}$ is a character of group H. We extend $\alpha_{p,q}$ from H to B_- by the rule: for

$$b_- = z_-h \in B_-,$$
$$z_- \in Z_-,$$
$$h \in H,$$
$$\alpha_{p,q}(b_-) = \alpha_{p,q}(z_-h) = \alpha_{p,q}(h).$$

For $(p, q) \in \mathbb{C}^2$, we define an induced representation $T^{\alpha_{p,q}} = ind_{B_-}^G \alpha_{p,q}$. It operates in the space

$$F_{\alpha_{p,q}}(G) = \{f \in C^\infty(G) | f(b_-g) = \alpha_{p,q}(b_-)f(g), b_- \in B_-, g \in G\}$$

where $C^\infty(G)$ is the set of all complex-valued smooth functions on G, by

$$S_g^{\alpha_{p,q}} f(h) = f(g^{-1}h), h, g \in G.$$

Because the subset B_-Z_+ is dense in G that is every point of G is a limit point of B_-Z_+ or a point of B_-Z_+ (Saenkarun, 2009), the functions from the space $F_{\alpha_{p,q}}(G)$ are completely determined by their restrictions to the subgroup Z_+. This allows to realize the representations $S^{\alpha_{p,q}}$ of G in the space $C^\infty(Z_+)$. The respective representation of the Lie algebra $\mathfrak{g}$ in the space $C^\infty(Z_+)$ is realized via the differential operators as follows: It will be convenient to introduce new parameters. Then we obtain

$$S_\alpha = -\frac{1}{h'(x)}\frac{\partial}{\partial x} - y\frac{\partial}{\partial z},$$

$$S_{\beta} = -\frac{\partial}{\partial y},$$
$$S_{-\alpha} = \frac{h^2(x)}{h'(x)}\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} - sh(x),$$
$$S_{-\beta} = \left(\frac{z - h(x)y}{h'(x)}\right)\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z} - ty.$$

Since Z_+ is a group with 3-dimensional nilpotent Lie algebra over $\mathbb{R}$, we obtain Z_+ is diffeomorphic to $\mathbb{R}^3$, i.e., there is a bijective map $\Phi: Z_+ \longrightarrow \mathbb{R}^3$ such that Φ and Φ^{-1} are smooth. Thus we can consider the subspace of V of all complex-valued smooth functions on $\mathbb{R}^3$ for the representation space of this representation of $\mathfrak{g}$.

Denote this representation by $\varphi^{s,t}$. Note that only 1 is annulled by the above positive root vectors, so 1, whose weight is $s\omega_1 + t\omega_2$, where ω_1, ω_2 are fundamental weights, is the only highest weight vector in V. Thus, for $\varphi^{s,t}$, where s, t are non-negative integers, applying products of $S_{-\alpha}$, $S_{-\beta}$ to 1, we obtain an invariant subspace of the irreducible representation of $\mathfrak{g}$.

For $s = 1, t = 0$, the subspace is spanned by 3 polynomials: $1, h(x), h(x)y - z$ and for $s = 0, t = 1$, the subspace is spanned by 3 polynomials: $1, y, z$.

Irreducible representations of A_2

Let $V^{s,t}$ be the real vector space spanned by

$$h^a(x)(h(x)y - z)^b y^c z^d,$$

where $a, b, c, d \geq 0$ and $a + b \leq s$, $c + d \leq t$.

By direct computation, we obtain $V^{s,t}$ is invariant under $\varphi^{s,t}$. Thus $\varphi^{s,t}$ is a finite-dimensional representation of $\mathfrak{g}$ in $V^{s,t}$. Since 1, whose weight is $s\omega_1 + t\omega_2$, is the only highest weight vector of $\varphi^{s,t}$ in $V^{s,t}$, and $\varphi^{s,t}$ is completely reducible, so $\varphi^{s,t}$ is an irreducible representation of $\mathfrak{g}$ in $V^{s,t}$ (Zhelobenko, 1973).

Solutions of the PDEs

As proposed by Saenkarun (2009), let us consider the space $C^\infty(G)$ two representations of the group G, $S_g f(h) = f(g^{-1}h)$ and $T_g f(h) = f(hg)$ where $h, g \in G$. Observe that S_g and T_h commute in $C^\infty(G)$ for all $g, h \in G$. Let $\mathfrak{X}$ be the dual space for the space $C^\infty(G)$, that is the space of all distributions with compact support on G. Consider in $\mathfrak{X}$ two representations of the group G conjugate to S_g and T_g that we shall denote by $\tilde{S}_g$ and $\tilde{T}_g$, where $\tilde{S}_g = S^*_{g^{-1}}$ and $\tilde{T}_g = T^*_{g^{-1}}$. Here $*$ denotes the adjoint operator, that is, if $\langle f, F\rangle$ is the canonical bilinear form for the pair $C^\infty(G)$ and $\mathfrak{X}$, then $\langle Af, F\rangle$=$\langle f, A^*F\rangle$ for any linear operator A in $C^\infty(G)$, $f \in C^\infty(G)$ and $F \in \mathfrak{X}$. Let 1_e be the δ-function on G with support at the identity e of the group G. Then it is easy to see that $\tilde{S}_g 1_e = 1_g$ and $\tilde{T}_g 1_e = 1_{g^{-1}}$, where 1_g is the δ-function with support at the point g of G and $1_{g^{-1}}$ is the same for g^{-1}. This implies that $\tilde{S}_g 1_e = \tilde{T}_{g^{-1}} 1_e$.

Because the representations S and T are C^∞-differentiable, we may consider their differentials, that is the representations of the universal enveloping algebra $U(\mathfrak{L})$, which we shall also denote by S and T, and the conjugate representations will be again denoted by $\tilde{S}$ and $\tilde{T}$. In algebra $U(\mathfrak{L})$, we consider the principal anti-automorphism $u \longrightarrow u'$ where $u' = -u$ for $u \in \mathfrak{L}$ and $(u_1 u_2 \dots u_k)' = u'_k u'_{k-1} \dots u'_1$. Then we get $\tilde{S}(u)1_e = \tilde{T}(u')1_e$ for all $u \in U(\mathfrak{L})$.

The space $E_\sigma = F_{\alpha_{p,q}}(G) = \{f \in C^\infty(G) | f(gb_-) = \alpha_{p,q}^{-1}(b_-)f(g), b_- \in B, g \in G\}$ can be identified as the space of the solutions to the system of PDE $T(x_\gamma)f = 0$, for all $\gamma \in \Pi_0$ (Π_0- the set of simple roots). $T(x - \langle \sigma - \rho, x \rangle)f = 0$, for all $x \in \mathfrak{H}$, σ is the signature of the inducing representation and ρ the half-sum of all positive roots (Zhelobenko, 1973).

Denote by I_σ the cyclic submodule in $U(\mathfrak{L})$-module $\mathfrak{X}$ generated by the elements $\tilde{S}(x_\gamma)1_e$, for all $\gamma \in \Pi_0$, $\tilde{S}(x - \langle \sigma - \rho, x \rangle)1_e$, for all $x \in \mathfrak{H}$.

Proposition 5.1 *E_σ is the orthogonal complement for I_σ with respect to the canonical bilinear form $\langle .,. \rangle$.*
Proof. Let $f \in E_\sigma$. Then

$$\left\langle f, \tilde{T}_g \tilde{T}_{x_\beta} 1_e \right\rangle = \left\langle T_{g^{-1}} f, \tilde{S}_{x'_\beta} 1_e \right\rangle = \left\langle S_{x_\beta} T_{g^{-1}} f, 1_e \right\rangle = \left\langle T_{g^{-1}} S_{x_\beta} f, 1_e \right\rangle = 0, \text{ for all } \beta \in \Pi_0,$$

and $\left\langle f, \tilde{T}_g \tilde{T}_{x - \langle \sigma - \rho, x \rangle} 1_e \right\rangle = \left\langle T_{g^{-1}} f, \tilde{S}_{-x - \langle \sigma - \rho, x \rangle} 1_e \right\rangle = \left\langle T_{g^{-1}} S_{x - \langle \sigma - \rho, x \rangle} f, 1_e \right\rangle = 0$, for all $x \in h$. So that $E_\sigma \subset (I_\sigma)^\perp$.

Reversely, let $\varphi \in (I_\sigma)^\perp$. Then $T_g \varphi \in (I_\sigma)^\perp$, and

$$0 = \left\langle T_{g^{-1}} \varphi, \tilde{T}_{x_\beta} 1_e \right\rangle = \left\langle T_{g^{-1}} S_{x_\beta} \varphi, 1_e \right\rangle = \left\langle S_{x_\beta} \varphi, \tilde{T}_g 1_e \right\rangle = \left\langle T_{x_\beta} \varphi, 1_g \right\rangle, \text{ for all } g \in G.$$

But this is equivalent to $S_{x_\beta} \varphi = 0$. Similarly for $\langle \sigma - \rho, x \rangle$, $x \in h$. Thus $E_\sigma = (I_\sigma)^\perp$. This is the end of the proof.

Let $\sigma \in \mathfrak{H}^*$ and χ be a positive root such that $\sigma(\chi) = N$, where N is a positive integer, and let M_σ be the Verma module corresponding to σ, and 1_σ is a highest weight vector of weight $\sigma - \rho$ in the module M_σ. Then $M_{\sigma - N\chi}$ is imbeddable into M_σ and so there exists $S_{\sigma,\chi}^N$ in the universal enveloping algebra of a maximal nilpotent subalgebra of $\mathfrak{L}$ spanned by all negative root vectors such that $\tilde{1}_{\sigma - N\chi} = S_{\sigma,\chi}^N 1_\sigma$, where $\tilde{1}_{\sigma - N\chi}$ is the image of $1_{\sigma - N\chi}$ under the imbedding (Loutsiouk, 2008).

Proposition 5.2 *$T(S_{\sigma,\chi}^N)$ is an intertwining operator for E_σ and $E_{\sigma'}$, $\sigma' = \sigma - N\chi$.*
Proof. It is sufficient to show that $S(T_{\sigma,\chi}^N)E_\sigma \subset E_{\sigma'}$. Let $f \in E_\sigma$ and $\gamma \in \prod_0$. Then

$$\left\langle S_{T_{\sigma,\chi}^N} f, \tilde{T}_g \tilde{T}_{x_\gamma} 1_e \right\rangle = \left\langle T_{g^{-1}} f, \tilde{T}_{x_\gamma} \tilde{S}_{(T_{\sigma,\chi}^N)'} 1_e \right\rangle = \left\langle T_{g^{-1}} f, \tilde{T}_{x_\gamma} \tilde{T}_{T_{\sigma,\chi}^N} 1_e \right\rangle = \left\langle T_{g^{-1}} f, \tilde{T}_{x_\gamma T_{\sigma,\chi}^N} 1_e \right\rangle.$$

But $x_\gamma T_{\sigma,\chi}^N \in I_\sigma$. So $\tilde{T}_{x_\gamma T_{\sigma,\chi}^N} 1_e \in I_\sigma$. Since $T_{g^{-1}} f \in E_\sigma$, the last expression is equal to 0.

Let $x \in \mathfrak{H}$. Then

$$\begin{aligned} \left\langle S_{T_{\sigma,\chi}^N} f, \tilde{T}_g \tilde{T}_{(x - \langle \sigma - N\chi - \rho, x \rangle)} 1_e \right\rangle &= \left\langle T_{g^{-1}} f, \tilde{S}_{(T_{\sigma,\chi}^N)'} \tilde{T}_{(x - \langle \sigma - N\chi - \rho, x \rangle)} 1_e \right\rangle \\ &= \left\langle T_{g^{-1}} f, \tilde{T}_{(x - \langle \sigma - N\chi - \rho, x \rangle) T_{\sigma,\chi}^N} 1_e \right\rangle. \end{aligned}$$

But $(x - \langle \sigma - N\chi - \rho, x \rangle) T_{\sigma,\chi}^N \in I_\sigma$. So $\tilde{T}_{(x - \langle \sigma - N\alpha - \rho, x \rangle) T_{\sigma,\chi}^N} 1_e \in I_\sigma$. Therefore the last expression equals 0, because $T_{g^{-1}} f \in E_\sigma \subset I_\sigma^\perp$. This is the end of the proof.

When γ is a simple root, then $S_{\sigma,\gamma}^{N} = z_{-\gamma}^{N}$, where for any root λ, Z_λ is a root vector for λ. For the simple root α,

$$I_1 f(h) = \frac{d}{da} T_{E_\alpha^{s+1}(a)} f(h)|_{a=0}$$

$$= \left(\frac{1}{h'(x)} \frac{\partial}{\partial x}\right)^{s+1} f(h)$$

$$= A^{s+1} f(h)$$

is an intertwining operator for $E_{s\omega_1+t\omega_2}$ and $E_{(-s-2)\omega_1+(s+t+1)\omega_2}$. So, $V^{s,t} \subseteq Ker\ \ I_1$.

For the simple root β,

$$I_2 f(h) = \frac{d}{da} T_{E_\beta^{t+1}(a)} f(h)|_{a=0}$$

$$= \left(\frac{\partial}{\partial y} + h(x) \frac{\partial}{\partial z}\right)^{t+1} f(h)$$

$$= B^{t+1} f(h)$$

is an intertwining operator for $E_{s\omega_1+t\omega_2}$ and $E_{(s+t+1)\omega_1+(-t-2)\omega_2}$. So $V^{s,t} \subseteq Ker\ \ I_2$.

Then $V^{s,t} = (Ker\ \ I_1) \cap (Ker\ \ I_2)$. That is, $h^a(x)(h(x)y - z)^b y^c z^d$, where $a, b, c, d \geq 0$ and $a + b \leq s$, $c + d \leq t$, are all solutions of the system (1), and they are the only solutions of this system of partial differential equations.

Conclusion

In this paper, we study a method to find all solutions of a certain systems of PDEs by considering representations of a Lie group A_2. We found that the space of solutions of the systems of partial differential equations

$$\left[\frac{1}{h'(x)} \frac{\partial}{\partial x}\right]^{s+1} \varphi = 0,$$
$$\left[\frac{\partial}{\partial y} + h(x) \frac{\partial}{\partial z}\right]^{t+1} \varphi = 0,$$

where s and t are non-negative integers, and the operators $\frac{1}{h'(x)} \frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y} + h(x) \frac{\partial}{\partial z}$ generate a Lie algebra of differential operators, that is isomorphic to a maximal nilpotent subalgebra of exceptional Lie algebra of class A_2, is the space $V^{s,t}$ of an irreducible representation spanned by the vectors $h^a(x)(h(x)y - z)^b y^c z^d$ where $a, b, c, d \geq 0$ and $a + b \leq s$, $c + d \leq t$.

For example, let $h(x) = x^3$ and $(s, t) = (1,0)$. Then the space of solutions of PDEs

$$
\begin{aligned}
\left[\frac{1}{3x^2}\frac{\partial}{\partial x}\right]^2 \varphi &= 0, \\
\left[\frac{\partial}{\partial y} + x^3\frac{\partial}{\partial z}\right]\varphi &= 0,
\end{aligned} \tag{2}
$$

is the space $V^{1,0}$ spanned by the vectors $1, x^3, x^3y - z$.

In the example, we may find all vectors that span the space of solutions of PDEs (2) $V^{1,0}$ by applying products of $S_{-\alpha} = \frac{(x^3)^2}{(3x^2)}\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} - (1)(x^3)$, $S_{-\beta} = \left(\frac{z-(x^3)y}{(3x^2)}\right)\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z} - (0)y$ to 1.

By the same method, we claim that the space of solutions of the systems of partial differential equations

$$
\begin{aligned}
\left[f(x,y,z)\frac{\partial}{\partial x}\right]^{s+1} \varphi &= 0, \\
\left[h(x,y,z)\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + g(x,y,z)\frac{\partial}{\partial z}\right]^{t+1} \varphi &= 0,
\end{aligned}
$$

where s and t are non-negative integers, $f(x,y,z) = \frac{1}{g_x(x,y,z)}$, and $h(x,y,z) = -\left(\frac{g(x,y,z)g_z(x,y,z)+g_y(x,y,z)}{g_x(x,y,z)}\right)$, is the space $V^{s,t}$ of an irreducible representation spanned by the vectors $g^a(x,y,z)(yg(x,y,z) - z)^b y^c z^d$ where $a, b, c, d \geq 0$ and $a + b \leq s$, $c + d \leq t$.

For example, let $g(x,y,z) = xyz$ and $(s,t) = (1,0)$. Then the space of solutions of PDEs

$$
\begin{aligned}
\left[\frac{1}{yz}\frac{\partial}{\partial x}\right]^2 \varphi &= 0, \\
\left[-\left(x^2y + \frac{x}{y}\right)\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + xyz\frac{\partial}{\partial z}\right]\varphi &= 0,
\end{aligned}
$$

is the space $V^{1,0}$ spanned by the vectors $1, xyz, xy^2z - z$.

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