



Lie Structure and Finite-Dimensional Simple Poisson Modules of two Poisson Algebras

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Abstract

This research classifies all finite-dimensional simple Poisson modules for two distinct Poisson algebras, $A = \mathbb{C}[x, y, z]$ and $B = \mathbb{C}[x, y, z]$, defined over the complex field $\mathbb{C}$. The Poisson brackets are explicitly defined as: $A: \{x, y\} = xy - x - y, \{y, z\} = yz - y - z, \{z, x\} = xz - x - z$ and $B: \{x, y\} = z^2 - 1, \{y, z\} = x, \{z, x\} = y$. Applying Jordan's classification framework (2010), we identify the Poisson maximal ideals, J , for both A and B , and analyze the structure of the associated Lie algebras, $\mathfrak{g}(J) = J/J^2$. We establish that Poisson algebras, A and B , admit exactly two Poisson maximal ideals, which are J_1 and J_2 . Crucially, we prove that for all four cases, the associated Lie algebras $\mathfrak{g}(J_i) = J_i/J_i^2$, $i = 1, 2$, are three-dimensional and isomorphic to the special linear algebra $\mathfrak{sl}(2, \mathbb{C})$. Consequently, utilizing the well-known representation theory of $\mathfrak{sl}(2, \mathbb{C})$, we show that the finite-dimensional simple Poisson modules of both A and B consist of two infinite families. For every dimension $d \geq 1$, there exists exactly one d – dimensional simple Poisson module (up to isomorphism) annihilated by J_1 and one by J_2 .

Keywords: Poisson algebras; Poisson ideal; Poisson maximal ideal; Derived algebra; Poisson modules

โครงสร้างลีและมอดูลพัลส์ของเชิงเตียมมิติจำกัดบนพีชคณิตพัลส์ของสองชนิด

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บทคัดย่อ

งานวิจัยนี้ทำการจำแนกมอดูลพัลส์ของเชิงเตียมมิติจำกัดทั้งหมดบนพีชคณิตพัลส์ของสองชนิดที่แตกต่างกันคือ $A = \mathbb{C}[x, y, z]$ และ $B = \mathbb{C}[x, y, z]$ ซึ่งนิยามบนฟิลด์จำนวนเชิงซ้อน $\mathbb{C}$ โดยวงเล็บพัลส์ของได้ให้นิยามไว้อย่างชัดเจนดังนี้ สำหรับพีชคณิตพัลส์ของ A : $\{x, y\} = xy - x - y$, $\{y, z\} = yz - y - z$, $\{z, x\} = xz - x - z$ และ B : $\{x, y\} = z^2 - 1$, $\{y, z\} = x$, $\{z, x\} = y$ โดยการประยุกต์ใช้กรอบแนวคิดการจำแนกของจอร์แดน (2010) เราได้ระบุไอดีลพัลส์ของใหญ่ที่สุด J ของพีชคณิตพัลส์ของทั้งสองชนิด พร้อมทั้งวิเคราะห์โครงสร้างของพีชคณิตลีที่เกี่ยวข้องคือ $\mathfrak{g}(J) = J/J^2$ ผลการวิจัยพบว่า พีชคณิตพัลส์ของ A และ B ต่างมีไอดีลพัลส์ของใหญ่ที่สุดสองค่าคือ J_1 และ J_2 ยิ่งไปกว่านั้นเราสามารถพิสูจน์ได้ว่าทั้งสกรณิพีชคณิตลี $\mathfrak{g}(J_i) = J_i/J_i^2$, $i = 1, 2$ ที่เกี่ยวข้องจะมีมิติเท่ากับ 3 และสมสัณฐานกับพีชคณิตลีเชิงเส้นพิเศษ $\mathfrak{sl}(2, \mathbb{C})$ ด้วยเหตุนี้เมื่อประยุกต์ใช้กับทฤษฎีการแทนที่รู้จักกันดีของ $\mathfrak{sl}(2, \mathbb{C})$ จึงแสดงให้เห็นว่ามอดูลพัลส์ของเชิงเตียมมิติจำกัดของทั้งสองพีชคณิตพัลส์ของ A และ B ประกอบด้วยชุดที่เป็นอนันต์จำนวนสองชุด กล่าวคือสำหรับทุกมิติ $d \geq 1$ จะมีมอดูลพัลส์ของเชิงเตียมมิติจำกัด d เพียงหนึ่งเดียวภายใต้การสมสัณฐานที่ถูกทำให้เป็นศูนย์โดย J_1 และอีกหนึ่งมอดูลที่ถูกให้เป็นศูนย์โดย J_2

คำสำคัญ: พีชคณิตพัลส์ของ; ไอดีลพัลส์ของ; ไอดีลพัลส์ของใหญ่ที่สุด; พีชคณิตอนุพัลส์; มอดูลพัลส์ของ

Introduction

A Poisson algebra A is a commutative $\mathbb{C}$ -algebra equipped with a Lie bracket $\{-, -\}$ satisfying the Leibniz rule: $\{ab, c\} = a\{b, c\} + \{a, c\}b$ for all $a, b, c \in A$. The bracket $\{-, -\}$ is then called a Poisson bracket. Poisson algebras play a central role in both algebraic structures and mathematical physics, as they unify commutative algebra and Lie theory in a single framework. Poisson modules, defined as modules compatible with both algebraic and Lie operations, provide further insight into the representation theory of these algebras.

The study of Poisson modules has gained significant attention in recent years due to its profound connections with various areas of mathematics and theoretical physics, particularly in deformation quantization and the representation theory of non-commutative algebras. A fundamental problem in this field is the classification of finite-dimensional simple Poisson modules over affine Poisson algebras. Early approaches by Sasom (2006) and Chen (2019) utilized direct methods for such classifications. Subsequently, the seminal work of Jordan (2010) established a systematic framework using the associated Lie algebras of Poisson maximal ideals. This breakthrough allowed researchers to explore diverse Poisson structures; for instance, Chansuriya and Fongchanta (2022) as well as Sasom and Changtong (2023) demonstrated that even slight modifications to the Poisson bracket can lead to significantly different Lie structures and module dimensions. Additionally, Xia and Lie (2022) extended representation theory into GIM Lie algebras, revealing structural constraints where non-trivial finite-dimensional simple modules do not exist in higher dimensions.

Moreover, several studies have contributed to the theory of Poisson modules. Oh (1999) introduced the notion of Poisson enveloping algebras, while Farkas (2000) investigated module structures for Poisson algebras. A significant development was made by Jordan (2010), who provided a systematic classification of finite-dimensional simple Poisson modules and established a direct relationship between Poisson maximal ideals and Lie modules.

Building on this framework, Changtong and Sasom (2018) as well as Sasom and Changtong (2023) studied specific Poisson algebras, successfully classifying their finite-dimensional simple Poisson modules. In particular, for a Poisson algebra $A' = \mathbb{C}[x, y, z]$, Changtong and Sasom (2018) revealed that for one of its maximal ideals, J_1 , the simple Poisson modules were restricted to be one-dimensional. The result suggests that the Lie algebra structure $\mathfrak{g}(J) := J/J^2$ of A was likely simple or abelian. However, minor alterations to the Poisson bracket can drastically change the $\mathfrak{g}(J)$ structure.

The aim of this study is to classify the modules for two new Poisson algebras, A and B . We demonstrate that the associated Lie algebras for both algebras are isomorphic to the simple Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. The crucial finding leads to a complete classification where simple modules exist in all dimensions $d \geq 1$, thus revealing a more complex and complete representation structure for this class of Poisson algebras compared to the previously studied A .

Preliminaries

This section contains some of the materials that will be used throughout this work. The main topics are Lie algebras Dixmier (1996), Derived algebras, Low-dimensional Lie algebras modules Erdmann and Wildon (2006) and Humphreys (1972), Poisson algebras and Poisson and Jordan's result shown in Jordan (2010).

Lie algebras

We begin by recalling the definition of a Lie algebra which leads to the definition of a Poisson algebra. Lie algebra arise as a vector space of linear transformations endowed with a new operation which is general neither commutative nor associative denoted by $[x, y] = xy - yx$ and called the Lie bracket or commutator of x and y .

Definition 1. Let F be a field. A **Lie algebra** over F is an F -vector space L , together with a bilinear map, the Lie bracket $[-, -] : L \times L \rightarrow L$ given by $(x, y) \mapsto [x, y]$ satisfying the following properties:

- 1) $[x, x] = 0$ for all $x \in L$,
- 2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$.

Axiom 2) is called the **Jacobi identity**. A Lie algebra **commutative** if $[x, y] = 0$ for all $x, y \in L$.

Definition 2. Let L be a Lie algebra. A subspace K of L is called a **Lie subalgebra** if $[x, y] \in K$ for all $x, y \in K$.

Definition 3. A subspace I of a Lie algebra L is called a **Lie ideal** of L if for all $x \in L$, $y \in I$, $[x, y] \in I$.

Definition 4. Let L and M be Lie algebras and a **Lie algebra homomorphism** is a linear map between to Lie algebras L and M such that it is compatible with the Lie bracket:

$$\theta : L \rightarrow M \text{ and } \theta([x, y]) = [\theta(x), \theta(y)] \text{ for all } x, y \in L.$$

Definition 5. A Lie algebra L is called **irreducible** if the only ideals in L are L and $\{0\}$. A Lie algebra L is called **simple** if it is irreducible and $\dim L \geq 2$.

Example 6. The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is identified with the Lie algebra of 2×2 matrices. We defined the following three elements

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

as a basis. The Lie products among e , f , and h are given by

$$[e, f] = h, \quad [h, e] = 2e \quad \text{and} \quad [h, f] = -2f.$$

It is well known that the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is 3-dimensional.

Proposition 7. (Hall, 2015) The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is simple.

Derived algebras

The derived algebra is one of the tools to investigate the results. Let I and J be ideals of a Lie algebra L , we define a product of ideals $[I, J]$ to be the span of the commutators of elements of I and J .

Definition 8. Let L be a Lie algebra over the complex field $\mathbb{C}$. The **derived algebra** (or **commutator algebra**) of L , denoted by $[L, L]$, is defined as the subspace of L is generated by all Lie brackets of elements of L . That is,

$$[L, L] := \text{Span}\{[x, y] : x, y \in L\}.$$

Evidently, L is **commutative** if and only if $[L, L] = 0$. Note that a one-dimensional Lie algebra is necessary commutative since $[ax, bx] = 0$ for all $x \in L$ and all scalars a and b .

Example 9. Let $\mathfrak{gl}(n, F)$ be the set of $n \times n$ matrices over a field F and $\mathfrak{sl}(n, \mathbb{C})$ be the subspace of $\mathfrak{gl}(n, F)$ consisting of all matrices of traces 0. Recall that the trace of a square matrix is the sum of its diagonal entries. Then $[\mathfrak{gl}(n, F), \mathfrak{gl}(n, F)] = \mathfrak{sl}(2, \mathbb{C})$ and $[\mathfrak{sl}(2, \mathbb{C}), \mathfrak{sl}(2, \mathbb{C})] = \mathfrak{sl}(2, \mathbb{C})$.

The following theorem is very useful for classifying the finite-dimensional simple Lie algebras.

Theorem 10. (Erdmann and Wildon, 2006) Suppose that L is a 3-dimensional complex Lie algebra such that $[L, L] = L$. Then $L \cong \mathfrak{sl}(2, \mathbb{C})$.

Poisson Algebras and Poisson modules

In this section, we present definition of Poisson algebras, Poisson modules and a Lie algebra J/J^2 , where J is a Poisson maximal ideal.

Definition 11. Let A be a commutative algebra over the complex field $\mathbb{C}$. A **Poisson bracket** on A is a Lie algebra bracket $\{-, -\}$ satisfying the Leibniz rule:

$$\{ab, c\} = a\{b, c\} + \{a, c\}b \text{ for all } a, b, c \in A.$$

The pair $(A, \{-, -\})$ is called a **Poisson algebra**.

Definition 12. Let A be a Poisson algebra over the complex field $\mathbb{C}$. A subspace B of A is a **Poisson subalgebra** of A if B is closed under both associative multiplication and the Poisson bracket of A , that is, for all $b, c \in B$, $xy \in B$ and $\{b, c\} \in B$.

Definition 13. An ideal I of a Poisson algebra A is called a **Poisson ideal** if $\{i, a\} \in I$ for all $i \in I$ and all $a \in A$. If I is a Poisson ideal of A then the quotient A/I becomes a Poisson algebra with the operation defined by:

$$\{a + I, b + I\} = \{a, b\} + I$$

for all $a, b \in A$.

A Poisson algebra A is said to be **simple** if its only Poisson ideals are 0 and A .

Definition 14. A Poisson ideal I of a Poisson algebra A is called a **maximal Poisson ideal** if J is a Poisson ideal and $I \not\subseteq J$ then $I = A$. An ideal I of a Poisson algebra A is said to be a **Poisson maximal ideal** if I is a maximal ideal of A and also a Poisson ideal.

Theorem 15. (Sharp, 2000) (Hilbert's Nullstellensatz Theorem) Let $\mathbb{F}[x_1, x_2, \dots, x_n]$ be such that polynomial ring over a field $\mathbb{F}$ in the indeterminates $x_1, x_2, \dots, x_n$. The ideal M is a maximal ideal if and only if there exist $a_1, a_2, \dots, a_n$ such that

$$M = (x_1 - a_1, x_2 - a_2, \dots, x_n - a_n).$$

Next, we present the definition of Poisson module. There is more than one definition of Poisson module in the literature. We shall use the one introduced by Farkas (2000).

Definition 16. Let A be a commutative Poisson algebra with Poisson bracket $\{-, -\}$. An A -module M is a **Poisson A -module** if there is a bilinear form $\{-, -\}_M: A \times M \rightarrow M$ such that the following axioms hold for all $a, b \in A$ and all $m \in M$:

$$(i) \{a, bm\}_M = \{a, b\}m + b\{a, m\}_M;$$

$$(ii) \{ab, m\}_M = a \{b, m\}_M + b \{a, m\}_M ;$$

$$(iii) \{\{a, b\}, m\}_M = \{a, \{b, m\}_M\}_M - \{b, \{a, m\}_M\}_M .$$

A submodule N of a Poisson module M is called a **Poisson submodule** if $\{a, n\}_M \in N$ for all $a \in A$ and $n \in N$.

Definition 17. Let N be a left module over a ring R . Give any subset $X \subseteq N$, the **annihilator** of X is the set $ann_R(X) = \{r \in R : rx = 0 \text{ for all } x \in X\}$, which is a left ideal of R .

Definition 18. Let M be a Poisson module over a Poisson algebra A and let $S \subseteq M$. In the module sense, we denote the annihilator of S in A by

$$ann_A(S) = \{r \in A : rx = 0 \text{ for all } x \in S\}$$

and we denote the Poisson annihilator of S in A by

$$Pann_A(S) = \{a \in A : \{a, m\}_M = 0 \text{ for all } m \in S\}.$$

Lemma 19. (Jordan, 2010) Let A be a Poisson algebra and M be a Poisson A -module. Let $J = ann_A(M)$. Then

1. J is a Poisson ideal of A .
2. If M is a finite-dimensional simple Poisson module then J is a Poisson maximal ideal of A .
3. $\mathbb{C} + J^2 \subseteq Pann_A M$.

Definition 20. Let L and M be Poisson algebras. An algebra homomorphism θ between Poisson algebras L and M is a **Poisson homomorphism** if it is also a Lie homomorphism.

Definition 21. An algebra homomorphism between two Poisson algebras is a **Poisson isomorphism** if it is a bijective Poisson homomorphism. Two Poisson brackets on the same algebra A are **equivalent** if the corresponding Poisson algebras are isomorphic. Let $(R, \{-, -\})$ be a Poisson algebra. We say that a $\mathbb{C}$ -algebra automorphism $\theta: R \rightarrow R$ is a **Poisson automorphism** if for all $x, y \in R$, $\theta(\{x, y\}) = \{\theta(x), \theta(y)\}$. Poisson automorphisms can be used to twist the module structures of a Poisson modules, as specified in the following theorem.

Definition 22. Let L and M be Poisson A -modules. A **Poisson module homomorphism** $\theta: L \rightarrow M$ is an A -module homomorphism such that $\theta(\{a, m\}_L) = \{a, \theta(m)\}_M$ for all $a \in A$ and $m \in M$. If a map θ is bijective then $\theta^{-1}: M \rightarrow L$ is also Poisson module homomorphism and θ is a Poisson module isomorphism.

Remarks 23. If $\theta: A \rightarrow B$ is a homomorphism of Lie algebras and M is a B -module then there is an A -module M^θ such that $M = M^\theta$, as a set, and $[a, m]_{M^\theta} = [\theta(a), m]_M$ for all $a \in A$ and all $m \in M$.

Lie algebra J/J^2 where J is a Poisson maximal ideal

Let A be Poisson algebra and let I and J be Poisson ideals of A . Then IJ is a Poisson ideal of A . Of course I and J are Lie subalgebras of A under the Poisson bracket $\{-, -\}$. If $I \subseteq J$, then I is a Lie ideal of J

and the quotient space J/I carries a Lie algebra structure. In particular, the quotient J/J^2 is always a Lie algebra, which we denoted by $\mathfrak{g}(J) := J/J^2$.

A natural approach to constructing Poisson modules is as follows: let I and J are Poisson ideals of a Poisson algebra A such that $I \subseteq J$, the quotient space J/I become a Poisson A/J -module with the Poisson bracket is denoted by $\{a, j + I\}_{J/I} = \{a, j\}_J + I$, for $a \in A$ and $j \in J$. It can be verified that this operation is well-defined and satisfies all Poisson module axioms. Under this construction, $\mathfrak{g}(J) = J/J^2$ also carries a Lie algebra structure. Since every Poisson A/J -submodule of $\mathfrak{g}(J)$ is a Lie ideal, it follows that if $\mathfrak{g}(J)$ is simple as a Lie algebra, it is also simple as a Poisson module. Furthermore, if A is an affine Poisson algebra and J is a Poisson maximal ideal such that $A/J \cong \mathbb{C}$, the converse also holds because every Lie ideal of $\mathfrak{g}(J)$ is then a Poisson A/J -submodule. We recall that an affine Poisson algebra is a Poisson algebra that is finitely generated as a $\mathbb{C}$ – algebra.

The following result by Jordan (2010) provides a fundamental framework for determining the finite-dimensional simple Poisson modules over any affine Poisson algebra. We utilize this theorem as a primary tool to investigate our main research problems. Specifically, the theorem establishes a one-to-one correspondence between finite-dimensional simple Poisson modules and finite-dimensional simple modules over a certain Lie algebra.

Theorem 24. (Jordan, 2010) Let A be an affine generated Poisson algebra.

(i) Let M be a finite-dimensional simple Poisson A -module and let $J = \text{ann}_A(M)$. There is a simple modules M^* for the Lie algebra $\mathfrak{g}(J)$ such that $M^* = M$, as $\mathbb{C}$ -vector space, and $[j + J^2, m]_{M^*} = \{j, m\}_M$ for all $j \in J$ and $m \in M$.

(ii) Let J be a Poisson maximal ideal of A and let N be a finite-dimensional simple $\mathfrak{g}(J)$ -module. There exists a simple Poisson A -module N' and a Lie homomorphism $\theta: A \rightarrow \mathfrak{g}(J)$ such that $N' = N^\theta$ as a Lie module over A and $J = \text{ann}_A(N')$.

(iii) For all finite-dimensional simple Poisson modules M , $M^{**} = M$. For all Poisson maximal ideals J of A and all finite-dimensional simple $\mathfrak{g}(J)$ -modules N , $N^{**} = N$.

(iv) The procedure in (i) and (ii) establish a bijection Γ from the set of isomorphism classes of finite-dimensional simple Poisson module over A to the set of pairs $(J, \hat{N})$, where J is a Poisson maximal ideal of A , N is a finite-dimensional simple $\mathfrak{g}(J)$ -module and $\hat{N}$ is the isomorphism class of N , given by $\Gamma(\hat{M}) = (\text{ann}_A(M), \hat{M}^*)$ where $\hat{M}$ is isomorphism class of M and $\hat{M}^*$ is isomorphism class of M^* .

Based on this correspondence, the classification of simple Poisson module for algebras A and B is achieved by identifying their Poisson maximal ideals and analyzing the associated Lie algebra structures, as presented in the following sections.

Results

In this section, we present the main results of this study, focusing on the determination of Poisson maximal ideals and the classification of their corresponding Lie algebras.

Theorem 25. Let $A = \mathbb{C}[x, y, z]$ be a commutative polynomial algebra generated by x, y, z . There are Poisson brackets on A given by

$$\{y, z\}_f = \frac{\partial f}{\partial x}, \quad \{x, y\}_f = \frac{\partial f}{\partial z}, \quad \{z, x\}_f = \frac{\partial f}{\partial y},$$

determined by polynomial functions $f := f(x, y, z) = xyz - xy - yz - zx \in A$ defined as follows;

$$\{x, y\} = xy - x - y, \quad \{y, z\} = yz - y - z, \text{ and } \{z, x\} = xz - x - z.$$

Proof. Let $f = xyz - xy - yz - zx \in A$ be the polynomial in A . Then the Poisson exact bracket are:

$$\begin{aligned}\{x, y\} &= \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(xyz - xy - yz - zx) = xy - x - y, \\ \{y, z\} &= \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xyz - xy - yz - zx) = yz - y - z, \\ \{z, x\} &= \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xyz - xy - yz - zx) = xz - x - z,\end{aligned}$$

Hence, these are the Poisson brackets on A :

$$\{x, y\} = xy - x - y, \quad \{y, z\} = yz - y - z, \quad \text{and} \quad \{z, x\} = xz - x - z.$$

In the next lemma, we find the Poisson maximal ideals of A for this Poisson bracket.

Lemma 26. In the above Poisson algebra A , there are only two Poisson maximal ideals of A which are the followings:

$$J_1 = xA + yA + zA \quad \text{and} \quad J_2 = (x - 2)A + (y - 2)A + (z - 2)A.$$

Proof. Let J be a Poisson maximal ideal of A . Since A is a commutative polynomial ring over the complex field $\mathbb{C}$, by Theorem 15, $J = (x - a, y - b, z - c)$ for suitable $a, b, c \in \mathbb{C}$. As J is a Poisson ideal, we have $\{x, J\} \in J$, $\{y, J\} \in J$, and $\{z, J\} \in J$. Observe that

$$\begin{aligned}yx - x - y &= \{x, y\} = \{x, y - b\} \in J, \\ zy - z - y &= \{y, z\} = \{y, z - c\} \in J, \\ xz - x - z &= \{z, x\} = \{z, x - a\} \in J.\end{aligned}$$

Then $ab - a - b = 0$, $bc - b - c = 0$, and $ac - a - c = 0$. This implies $ab - a - b + 1 = 1$ and so $(a - 1)(b - 1) = 1$. Similarly, we obtain $(a - 1)(c - 1) = 1$ and $(b - 1)(c - 1) = 1$. Thus $(a - 1) = (b - 1) = (c - 1) = 1$ or $(a - 1) = (b - 1) = (c - 1) = -1$. If $(a - 1) = (b - 1) = (c - 1) = 1$ then $a = b = c = 2$, and if $(a - 1) = (b - 1) = (c - 1) = -1$, we obtain $a = b = c = 0$. Therefore there are only two Poisson maximal ideals:

$$J_1 = xA + yA + zA \quad \text{and} \quad J_2 = (x - 2)A + (y - 2)A + (z - 2)A.$$

Theorem 27. The Lie algebra $\mathfrak{g}(J_1)$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$, where $J_1 = xA + yA + zA$ the Poisson maximal ideal of A .

Proof. For $J_1 = xA + yA + zA$, we consider the associated Lie algebra $\mathfrak{g}(J_1) = J_1/J_1^2$. In A , the Poisson brackets are given by

$$\begin{aligned}\{x, y\} &= xy - x - y, \\ \{y, z\} &= yz - y - z, \\ \{z, x\} &= xz - x - z.\end{aligned}$$

The Lie algebra $\mathfrak{g}(J_1)$ has a basis consisting of the images of x, y, z denoted by $\bar{x}, \bar{y}, \bar{z}$. Since $x, y, z \in J_1$, we have $xy, yz, zx \in J_1^2$. Therefore, in the quotient $\mathfrak{g}(J_1) = J_1/J_1^2$, these quadratic terms vanish. The Lie brackets in $\mathfrak{g}(J_1)$ are thus computed as

$$\begin{aligned} [\bar{x}, \bar{y}] &= -\bar{x} - \bar{y}, \\ [\bar{y}, \bar{z}] &= -\bar{y} - \bar{z}, \\ [\bar{z}, \bar{x}] &= -\bar{z} - \bar{x}. \end{aligned}$$

Next, we show that $[\bar{x}, \bar{y}]$, $[\bar{y}, \bar{z}]$, $[\bar{z}, \bar{x}]$ are linearly independent. Let a, b, c be such that $a[\bar{x}, \bar{y}] + b[\bar{y}, \bar{z}] + c[\bar{z}, \bar{x}] = 0$. Substituting the Lie brackets, we obtain $a(-\bar{x} - \bar{y}) + b(-\bar{y} - \bar{z}) + c(-\bar{z} - \bar{x}) = 0$, which simplifies to $(-a - c)\bar{x} + (-a - b)\bar{y} + (-b - c)\bar{z} = 0$. Since $\bar{x}, \bar{y}, \bar{z}$ are linearly independent, it follows that $a = b = c = 0$, confirming the linearly independence of the brackets. Therefore $[\mathfrak{g}(J_1), \mathfrak{g}(J_1)] = \text{span}\{[\bar{x}, \bar{y}], [\bar{y}, \bar{z}], [\bar{z}, \bar{x}]\}$ is 3-dimensional and $[\bar{x}, \bar{y}]$, $[\bar{y}, \bar{z}]$, $[\bar{z}, \bar{x}]$ are linearly independent. Consequently, $\mathfrak{g}(J_1)$ is a 3-dimensional Lie algebra, implying that the derived algebra $[\mathfrak{g}(J_1), \mathfrak{g}(J_1)] = \mathfrak{g}(J_1)$. By Theorem 10, $\mathfrak{g}(J_1)$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

Theorem 28. The Lie algebra $\mathfrak{g}(J_2)$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$, where $J_2 = (x - 2)A + (y - 2)A + (z - 2)A$ is the Poisson maximal ideal of A .

Proof. For $J_2 = (x - 2)A + (y - 2)A + (z - 2)A$, we consider the associated Lie algebra $\mathfrak{g}(J_2)$. Let $u = x - 2$, $v = y - 2$ and $w = z - 2$. Then $J_2 = uA + vA + wA$. In A , we compute the Poisson brackets as follows $\{u, v\} = \{x - 2, y - 2\} = \{x, y\} = xy - y - x = (u + 2)(v + 2) - (u + 2) - (v + 2) = uv + u + v$. Similarly, $\{v, w\} = vw + v + w$, and $\{w, u\} = uw + u + w$. The Lie algebra $\mathfrak{g}(J_2)$ has a basis consisting of the images of u, v , and w , denoted by $\bar{u}, \bar{v}, \bar{w}$. Since $u, v, w \in J_2$, their products $uv, vw, uw \in J_2^2$ and thus vanish in the quotient $\mathfrak{g}(J_2)$. The Lie brackets in $\mathfrak{g}(J_2)$ are

$$[\bar{u}, \bar{v}] = \bar{u} + \bar{v}, \quad [\bar{v}, \bar{w}] = \bar{v} + \bar{w}, \quad [\bar{w}, \bar{u}] = \bar{u} + \bar{w}.$$

To verify linear independence, let $a, b, c \in \mathbb{C}$ be such that $a[\bar{u}, \bar{v}] + b[\bar{v}, \bar{w}] + c[\bar{w}, \bar{u}] = 0$. Substituting the Lie brackets, we obtain $a(\bar{u} + \bar{v}) + b(\bar{v} + \bar{w}) + c(\bar{u} + \bar{w}) = 0$, which simplifies to $(a + b)\bar{u} + (a + b)\bar{v} + (b + c)\bar{w} = 0$. Since $\bar{u}, \bar{v}, \bar{w}$ are linearly independent, we have $a + b = 0$, $a + b = 0$ and $b + c = 0$, it follows that $a = b = c = 0$. Therefore the derived algebra $[\mathfrak{g}(J_2), \mathfrak{g}(J_2)] = \text{span}\{[\bar{u}, \bar{v}], [\bar{v}, \bar{w}], [\bar{w}, \bar{u}]\}$ is 3-dimensional and $[\bar{u}, \bar{v}]$, $[\bar{v}, \bar{w}]$, $[\bar{w}, \bar{u}]$ are linearly independent. This implies that the derived algebra $[\mathfrak{g}(J_2), \mathfrak{g}(J_2)] = \mathfrak{g}(J_2)$. Since $\mathfrak{g}(J_2)$ is a 3-dimensional complex Lie algebra, it follows from Theorem 10 that $\mathfrak{g}(J_2)$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

Corollary 29. The finite-dimensional simple Poisson modules of the Poisson algebra A are classified by the finite-dimensional simple modules of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. Consequently, for every dimension $d \geq 1$, there exists exactly two non-isomorphic simple Poisson modules of dimension d (one for each Poisson maximal ideal J_1 and J_2).

Proof. By Theorem 24, each finite-dimensional simple Poisson module is determined by a Poisson maximal ideal together with a finite-dimensional simple module over the associated Lie algebra. Since both J_1 and J_2 yield Lie algebras isomorphic to $\mathfrak{sl}(2, \mathbb{C})$, the classification of finite-dimensional simple modules over A reduces to the well-known representation theory of $\mathfrak{sl}(2, \mathbb{C})$.

Theorem 30. Let $B = \mathbb{C}[x, y, z]$ be a commutative polynomial algebra generated by x, y, z . There are Poisson brackets on B defined by polynomial $f(x, y, z) = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^3}{3} - z$,

$$\{y, z\}_f = \frac{\partial f}{\partial x}, \quad \{x, y\}_f = \frac{\partial f}{\partial z}, \quad \{z, x\}_f = \frac{\partial f}{\partial y}, \quad \text{and so}$$

$$\{x, y\} = z^2 - 1, \quad \{y, z\} = x, \quad \text{and} \quad \{z, x\} = y.$$

Then B is a Poisson algebra.

Proof. It is straightforward to verify that the bracket defined above is bilinear and antisymmetric. Direct computation shows that the Jacobi identity $\{x, \{y, z\}\} + \{y, \{z, x\}\} + \{z, \{x, y\}\} = 0$ holds. Furthermore, the Leibniz rule holds because the bracket is extended as a derivation from its values on the generators. That is, for each fixed $f \in B$, the map $g \mapsto \{f, g\}$ is a derivation. Thus B forms a Poisson algebra under this structure.

Lemma 31. In the Poisson algebra $B = \mathbb{C}[x, y, z]$ with the bracket defined above, there exist exactly two Poisson maximal ideals:

$$J_1 = xB + yB + (z - 1)B \quad \text{and} \quad J_2 = xB + yB + (z + 1)B.$$

Proof. Let J be a Poisson maximal ideal of B . Since B is a commutative polynomial ring over the complex field $\mathbb{C}$, by Theorem 15, $J = (x - a, y - b, z - c)$ for suitable $a, b, c \in \mathbb{C}$. As J is a Poisson ideal, we have $\{x, J\} \in J$, $\{y, J\} \in J$, and $\{z, J\} \in J$. Observe that

$$z^2 - 1 = \{x, y\} = \{x, y - b\} \in J,$$

$$x = \{y, z\} = \{y, z - c\} \in J,$$

$$y = \{z, x\} = \{z, x - a\} \in J.$$

Evaluating these at the point (a, b, c) , we obtain $c^2 - 1 = 0$, $a = 0$, $b = 0$. This implies $c = \pm 1$, while $a = 0$, $b = 0$. Therefore there exist exactly two Poisson maximal ideals:

$$J_1 = xB + yB + (z - 1)B \quad \text{and} \quad J_2 = xB + yB + (z + 1)B.$$

Theorem 32. Let J_1 be a Poisson maximal ideal of B . Then the Lie algebra $\mathfrak{g}(J_1) = J_1/J_1^2$ is 3-dimensional and isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

Proof. For the Poisson maximal ideal $J_1 = xB + yB + (z - 1)B$, we consider the associated Lie algebra $\mathfrak{g}(J_1) = J_1/J_1^2$. The basis of this 3-dimensional Lie algebra is given by images of the generators $u = x$, $v = y$, $w = z - 1$ in $\mathfrak{g}(J_1)$, denoted by $\bar{u}, \bar{v}, \bar{w}$. Since $z = w + 1$, we note that $\bar{z} = \bar{w} + 1$ and $\bar{w}^2 = 0$ in $\mathfrak{g}(J_1) = J_1/J_1^2$. We compute the Lie brackets using the rule $[\bar{f}, \bar{g}] = \overline{\{f, g\}}$. Consider

$$[\bar{u}, \bar{v}] = \overline{\{x, y\}} = \overline{z^2 - 1} = \overline{(w + 1)^2 - 1} = \overline{w^2 + 2w} = 2\bar{w},$$

$$[\bar{v}, \bar{w}] = \overline{\{y, z - 1\}} = \bar{x} = \bar{u}.$$

Similarly, we have $[\bar{w}, \bar{u}] = \bar{v}$. This shows that the derived algebra $[\mathfrak{g}(J_1), \mathfrak{g}(J_1)]$ is spanned by $\{\bar{u}, \bar{v}, 2\bar{w}\}$ which is the entire algebra $\mathfrak{g}(J_1)$. Since the Lie algebra $\mathfrak{g}(J_1)$ is 3-dimensional and non-commutative with $[\mathfrak{g}(J_1), \mathfrak{g}(J_1)] = \mathfrak{g}(J_1)$, it satisfies the conditions for unique simple Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, by Theorem 10, it follows that $\mathfrak{g}(J_1)$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

Theorem 33. Let J_2 be a Poisson maximal ideal of B . Then the Lie algebra $\mathfrak{g}(J_2) = J_2/J_2^2$ is three-dimensional and isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

Proof. For the Poisson maximal ideal $J_2 = xB + yB + (z + 1)B$, we consider the associated Lie algebra $\mathfrak{g}(J_2) = J_2/J_2^2$. The basis of this 3-dimensional Lie algebra is given by images of the generators $u = x$, $v = y$, $w = z + 1$ in $\mathfrak{g}(J_2)$, denoted by $\bar{u}, \bar{v}, \bar{w}$. Since $z = w - 1$, we note that $\bar{z} = \bar{w} - 1$ and $\bar{w}^2 = 0$ in $\mathfrak{g}(J_2) = J_2/J_2^2$. We compute the Lie brackets using the rule $[\bar{f}, \bar{g}] = \overline{\{f, g\}}$. Consider

$$[\bar{u}, \bar{v}] = \overline{\{x, y\}} = \overline{z^2 - 1} = \overline{(w - 1)^2 - 1} = \overline{w^2 - 2w} = -2\bar{w}, \quad [\bar{v}, \bar{w}] = \overline{\{y, z + 1\}} = \overline{\{y, z\}} = \bar{x} = \bar{u}.$$

Similarly, we have $[\bar{w}, \bar{u}] = \bar{v}$. The derived algebra $[\mathfrak{g}(J_2), \mathfrak{g}(J_2)]$ is spanned by $\{\bar{u}, \bar{v}, -2\bar{w}\}$ which spans $\mathfrak{g}(J_2)$. Since $[\mathfrak{g}(J_2), \mathfrak{g}(J_2)] = \mathfrak{g}(J_2)$ and $\mathfrak{g}(J_2)$ is 3-dimensional complex Lie algebra, by Theorem 10, it follows that $\mathfrak{g}(J_2)$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

Corollary 34. Every finite-dimensional simple Poisson module over B corresponds to a finite-dimensional simple modules over $\mathfrak{sl}(2, \mathbb{C})$.

Proof. By Theorem 24, each finite-dimensional simple Poisson module is determined by a Poisson maximal ideal together with a finite-dimensional simple module over the associated Lie algebra. Since both J_1 and J_2 yield Lie algebras isomorphic to $\mathfrak{sl}(2, \mathbb{C})$, the classification of finite-dimensional simple modules over B reduces to the well-known classification of simple $\mathfrak{sl}(2, \mathbb{C})$ -modules.

Conclusion and Discussion

This research successfully classified the finite-dimensional simple Poisson modules for two distinct Poisson algebras, $A = \mathbb{C}[x, y, z]$ and $B = \mathbb{C}[x, y, z]$, using the method proposed by Jordan (2010). The findings confirm that both Poisson algebras under investigation possess exactly two Poisson maximal ideals J_1 and J_2 . Crucially, the associated quotient Lie algebras, $\mathfrak{g}(J_i) = J_i/J_i^2$, for all four cases, are proven to be isomorphic to the simple Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. The $\mathfrak{sl}(2, \mathbb{C})$ structure leads directly to the complete classification the finite-dimensional simple Poisson modules of both A and B are shown to consist of two infinite families, this means that for every dimension $d \geq 1$, there exists exactly one simple module (up to isomorphism) annihilated by J_1 and another by J_2 . The most significant finding of this study is revealed when comparing the result of the current Poisson algebra A with a closely related algebra A' studied previously Changtong and Sasom (2018). In the prior work on A' , the modules annihilated by the corresponding J_1 were restricted to be only one-dimensional. The sharp contrast demonstrates that a seemingly minor modification to the Poisson bracket fundamentally alters the structure of the associated Lie algebra $\mathfrak{g}(J_i)$. Specifically, the bracket in the current algebra A results in highly complex and richer $\mathfrak{sl}(2, \mathbb{C})$ Lie structure for $\mathfrak{g}(J_i)$, which facilitates the existence of simple modules in all dimensions. This highlights the high sensitivity of the Poisson maximal ideal structure to the bracket definition. The classification of modules for both A and the classical-bracket algebra B , and their convergence to the $\mathfrak{sl}(2, \mathbb{C})$ structure, provides a substantial extension to the existing literature on Poisson module classification.

This framework suggests potential extensions to more general cases, such as Poisson algebra with more than three generators or those defined by more intricate Poisson bracket relations. Investigating such cases could reveal deeper algebraic phenomena where the associated Lie algebras may be of a higher dimension or isomorphic to other types of simple Lie algebras, providing a broader understanding of module classification in Poisson algebras.

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Research Ethics

This study did not involve research on humans, animals, or plants.

The use of Artificial Intelligence tools

The project team used Open AI's Chat GPT-4 and Gemini during the proposal drafting process to enhance the clarity and coherence of the language. All scientific content was independently developed by the team. We rigorously verified the accuracy and originality of the work, upholding established ethical standards for the responsible use of AI in research.

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