Exploring Properties of Quadrilaterals in Elliptic Geometry using the Dynamic Geometry Software
การสำรวจสมบัติของรูปสี่ด้านในเรขาคณิตอิลลิปติก โดยใช้ซอฟต์แวร์เรขาคณิตพลวัต

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Abstract

The purpose of this research was to demonstrate how students explore the important properties of elliptic quadrilaterals by using the Dynamic Geometry Software (DGS). The participants comprised 26 mathematics students in the fourth year of their undergraduate program in the Faculty of Education at Rajabhat Rajanagarindra University, Thailand. They had enrolled in the Foundations of Geometry course at the first semester of the academic year 2019. The instruments were activity packages exploring properties of Saccheri and Lambert quadrilaterals in elliptic geometry using DGS. The results indicated that the students could make conjectures and verify properties of elliptic quadrilaterals correctly and rapidly. The students concluded that the summit angles in a Saccheri quadrilateral are always congruent and obtuse. The line joining the midpoints of the base and summit of a Saccheri quadrilateral is perpendicular to both the base and the summit. They also concluded that in elliptic geometry, a Lambert quadrilateral has its fourth

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angle obtuse, and each side of this angle is shorter than the side opposite. Therefore, the use of DGS can help students visualize this non–Euclidean geometry.

Keywords: Saccheri quadrilateral, Lambert quadrilateral, elliptic geometry, dynamic geometry software

บทคัดย่อ

การวิจัยครั้งนี้มีวัตถุประสงค์เพื่อศึกษาวิธีการสำรวจสมบัติที่สำคัญของรูปสี่ด้านในเรขาคณิตอิลลิปติกของนักศึกษาโดยใช้ซอฟต์แวร์เรขาคณิตพลวัต กลุ่มตัวอย่างเป็นนักศึกษาคณะครุศาสตร์ สาขาวิชาคณิตศาสตร์ชั้นปีที่ 4 มหาวิทยาลัยราชภัฏราชนครินทร์ ประเทศไทย จำนวน 26 คน ที่ลงทะเบียนเรียนรายวิชารากฐานเรขาคณิตในภาคการศึกษาที่ 1 ปีการศึกษา 2562 เครื่องมือที่ใช้ในการวิจัยคือ ชุดกิจกรรมที่ใช้สำรวจสมบัติของรูปสี่ด้านแซคเคอรีและรูปสี่ด้านลัมแบร์ทในเรขาคณิตอิลลิปติกโดยใช้ซอฟต์แวร์เรขาคณิตพลวัต ผลการวิจัยพบว่านักศึกษาสามารถตั้งข้อความคาดการณ์และตรวจสอบสมบัติของรูปสี่ด้านแซคเคอรีและรูปสี่ด้านลัมแบร์ทได้ถูกต้องและรวดเร็ว นักศึกษาสรุปได้ว่ามุมซัมมิทของรูปสี่ด้านแซคเคอรีเป็นมุมป้านและมีขนาดเท่ากันเสมอ เส้นที่เชื่อมจุดกึ่งกลางของด้านฐานและด้านซัมมิทของรูปสี่ด้านแซคเคอรีจะตั้งฉากกับทั้งด้านฐานและด้านซัมมิท นักศึกษายังสรุปได้อีกว่าในเรขาคณิตอิลลิปติก รูปสี่ด้านลัมแบร์ทจะมีมุมที่สี่ (มุมซึ่งไม่เป็นมุมฉาก) เป็นมุมป้าน และด้านประชิดมุมที่สี่มีความยาวน้อยกว่าด้านตรงข้ามมุมที่สี่ ดังนั้น การใช้ซอฟต์แวร์เรขาคณิตพลวัตสามารถช่วยให้นักศึกษาค้นพบสมบัติที่สำคัญของเรขาคณิตนอกแบบยุคลิดประเภทนี้ได้

คำสำคัญ: รูปสี่ด้านแซคเคอรี, รูปสี่ด้านลัมแบร์ท, เรขาคณิตอิลลิปติก, ซอฟต์แวร์เรขาคณิตพลวัต

1. Introduction

Geometry is a classic mathematics subject. The word "geometry" is derived from two words "earth" (geo) and "measure" (metry). The idea of earth measure was significant in the ancient, pre–Greek development of geometry (Smart, 1998, p.1). It was originally the science of measuring land (Greenberg, 1993). Geometry had the origin and developments in classical times and most students in high schools are introduced to study geometry (Lezark & Capaldi, 2016). Geometry, of all of the branches of mathematics, has been most subject to changing tastes from age to age (Merzbach & Boyer, 2011). It is a natural outgrowth of our exposure to the physical universe and in particular to the natural world (Hvidsten, 2017). The role of geometry in education and daily life is enormous. Geometric shapes very often are real works of art. Learning solid geometry is important for its applications in physics, chemistry etc. Therefore, in particular, geometry is a powerful tool to attract students to mathematics (Dolbilin, 2004).

In Thailand, non–Euclidean geometries (hyperbolic or elliptic geometry) are covered on at the university level for students majoring in mathematics or mathematics education. Therefore, many mathematics teachers are unfamiliar with non-Euclidean geometry due to the fact that Euclidean geometry is the mainstream geometry taught at the primary and secondary levels (Buda, 2017). At the college level, geometry is still a difficult course for most students because it requires them to reason strictly from axioms, postulates and theorems rather than informal experiences and intuitive understandings. In order to enable students to appreciate the importance of the rigorous axiomatic approach, most college geometry courses introduce students to a less intuitive world of non–Euclidean geometry. Generally, students enter a college geometry course with twelve or more years of experience working within the Euclidean system of axioms. By this, students' understanding of figures and relationships within this system is challenged when the axioms are modified (Smith, Hollebrands, Iwancio & Kogan, 2007). While geometry is a very visual subject, there are several limitations to students' uses of paper–and–pencil diagrams, especially when it comes to non–Euclidean geometries. A student may create inaccurate misleading diagrams and arrive to incorrect conjectures. Also, a student may create a correct diagram that is too specific; this may inhabit students' ability to derive general conclusions and proofs that go beyond the drawing they have created (Schoenfeld, 1986).

The discovery of non–Euclidean geometry is one of the most important events in the history of mathematics. Not long after the development of hyperbolic geometry, the German mathematician Riemann (1826–1866) suggested a geometry, now called elliptic, based on the alternative to the fifth postulate in Euclidean geometry, which states that there are no parallels to a line through a point on the line or any two lines in a plane meet at an ordinary point.

A model for geometry is an interpretation of the technical terms of the geometry (such as point, line, distance, angle measure etc.) that is consistent with the axioms of the geometry (Venema, 2003). There are many ways in which models of elliptic geometry is constructed. Some of these models are the Stereo Graphic Projection model and the Sphere X-Y Projection model. This study was conducted by using elliptic geometry and the Stereo Graphic Projection model.

Historically, mathematicians have attempted to prove Euclid's fifth postulate (or, equivalently, Playfair's Axiom) as a theorem solely on the basis of the first four postulates. One mathematician, Giovanni Girolamo Saccheri (1667–1733) did not try to prove the fifth postulate directly, but instead tried to prove it by the method of contradiction. He looked at special figures in the plane which were now called Saccheri Quadrilaterals. Saccheri quadrilaterals are quadrilaterals whose base angles are right angles and whose base-adjacent sides are congruent. That is, the top (or summit) angles must be right angles. (Hvidsten, 2017). Johann Heinrich Lambert (1728–1777), like Saccheri, attempted to prove the fifth postulate by an indirect argument. He began with a quadrilateral with three right angles, now called a Lambert quadrilateral. Of course, in the Euclidean geometry a Saccheri or a Lambert quadrilateral has to be a rectangle, but the elliptic world is different.

Many mathematics educators, researchers, and professional organizations have suggested the use of dynamic geometry software (DGS) to help teaching geometry e.g. Geometer's Sketchpad, GeoGebra, Cabri. These software programs enable students to construct creatively an accurate diagram and to interact with the diagrams in order to abstract general properties and relationship because the ways in which the programs respond to the students' actions is determined by geometrical theorems. 'Dragging' feature of DGS distinguishes it from other geometry software (Goldenberg & Couco, 1998). After a construction is completed, the user can drag certain elements of it, and the whole construction behaves in such a way that specified constraints are maintained. This feature allows students to quickly and easily investigate the truth of a particular conjecture. These programs facilitate explorations that promote the conjecturing process (Guven & Karatas, 2009).

Geometry Explorer is designed by Michael Hvidsten. It is designed as a geometry laboratory where one can create geometric objects (like points, circles, polygons, areas, and the like, carry out transformations on these objects (dilations, reflections, rotations, and translations), and measure aspects of these objects (like length, area, radius, and so on). In this case, it is much like doing geometry on paper (or sand) with a ruler and compass. However, on paper such constructions are static–points placed on the paper can never be moved again. In Geometry Explorer, all constructions are dynamic. One can draw a segment and then grab one of the endpoints and move it around the canvas with the segment moving accordingly. Thus, one can create a construction and test out hypotheses about construction with numerous variations of the original construction. Geometry Explorer is just what the name implies–an environment to explore geometry (Hvidsten, 2005). A screenshot of the program for elliptic geometry can be seen in Figure 1.

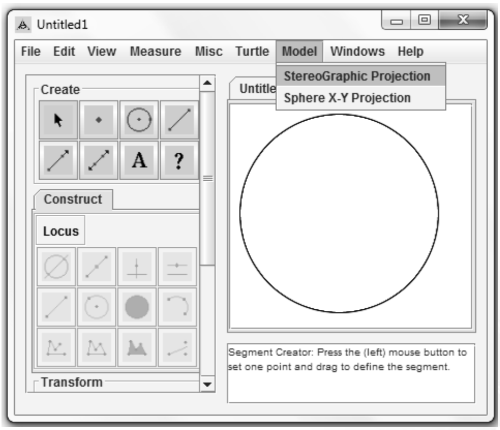


Figure 1: A screenshot of Geometry Explorer.

In the past, there had been many researches dedicated to studying the effect of DGS on students' progress along with their attitudes in geometry. Most of them emphasized that the use of DGS improved students' achievement, interest and participation in geometry (Groman, 1996; Bielefeld, 2002; Singmuang & Phahanich, 2004; Dogan & Icel, 2011; Erbas & Yenmez, 2011; Kurtuluş & Ada, 2011; Guven, 2012; Singmuang, 2013; Bhagat & Chang, 2015; Lorsong & Singmuang, 2015; Singmuang, 2016; Sebial, 2017; Singmuang, 2018).

2. Purpose of the Study

Evidently, diverse technological tools have been developed to facilitate students in reasoning within different non–Euclidean geometries such as Geometers' sketchpad, NonEuclid, GeoGebra, Cabri, but little research has examined how students' uses of the Geometry Explorer affects their understandings of properties of quadrilaterals in elliptic geometry. Therefore, this study was aimed to demonstrate how students who majoring in mathematics explore some of the properties of quadrilaterals in elliptic geometry by using the Geometry Explorer program.

3. Materials and Methods

The participants in this study were 26 students majoring in mathematics. These students were in the fourth year of their undergraduate program in the Faculty of Education at Rajabhat Rajanagarindra University, Thailand. They had enrolled in the Foundations of Geometry course at the first semester of the academic year 2019. These students were selected by purposive sampling. They were also volunteered to participate in this study.

This research arose from my experience in teaching the foundations of geometry course to students majoring in mathematics at Rajabhat Rajanagarindra University. The aim of this course is to enable students to acquire axiomatic nature, basic concepts and theorems of non-Euclidean geometry, development of hyperbolic geometry, development of elliptic geometry, development of spherical geometry, and development of projective geometry. Elliptic geometry was introduced during week 10–11 of the course. The Geometry Explorer program could allow students to have experience with this type of geometry.

This study emerged from my classroom observations while students were exploring the elliptic geometry with the Geometry Explorer during week 10–11 of the course. The observations in this study involved six lesson hours. The role of the teacher, an author of this article, was to help students explore elliptic geometry. Furthermore, the researcher made some observations focused on the students' conjectures and discussions. She noted important observations. In addition, students' worksheets were collected as data. The data were analyzed using the triad 'experiment–conjecture–explanation. By this way, the researcher aimed to determine the potential of DGS for studying the elliptic geometry.

The research instruments used in this study were activity packages exploring elliptic geometry by using the Geometry Explorer program. These packages were developed by the researcher. The instruments were pilot tested by administering to the students who were not part of the target population. The research activities were prepared under the subjects headings Saccheri quadrilaterals and Lambert quadrilaterls. The students were asked to explore the following properties:

1) The summit angles in a Saccheri quadrilateral are congruent.

2) The summit angles in a Saccheri quadrilateral are obtuse.

3) The line joining the midpoints of the base and summit of a Saccheri quadrilateral is perpendicular to the base and the summit.

4) In a Lambert quadrilateral the fourth angle (the one that is not a right angle) is always obtuse.

5) In a Lambert quadrilateral, each side adjacent to the fourth angle (the one that is not a right angle) has length shorter than the opposite side.

After being introduced to the technical properties of Geometry Explorer software for an hour, the sampled students were introduced the basic concepts of elliptic geometry, definition of a Saccheri quadrilateral, definition of a Lambert quadrilateral by using the Geometry Explorer for 2 hours. Then, they were asked to complete the activities by using Geometry Explorer tools so that they explore the elliptic geometry modeled by the Stereo Graphic Projection model for 3 hours.

4. Results

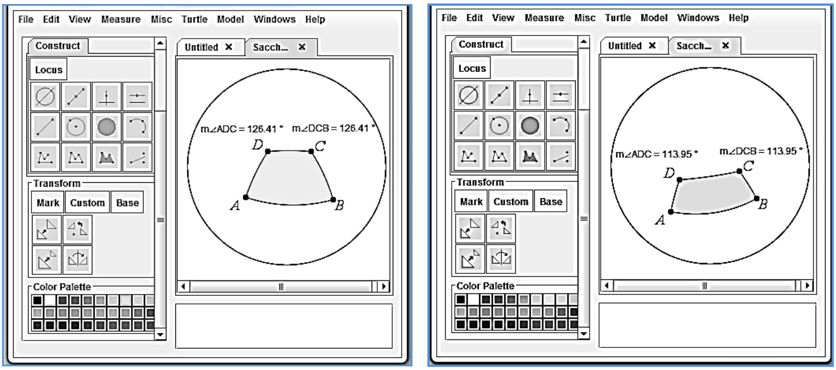
After introducing the basic concepts of elliptic geometry, such as point, line, angle, circle, perpendicular line, definition of a Saccheri quadrilateral, definition of a Lambert quadrilateral to the students by using the Geometry Explorer, The students began their exploration activities in the computer–based environment.

The ways students used the Dynamic Geometry Software (DGS) as they explored properties of elliptic quadrilaterals in each activity were as follows:

Exploration 1: The summit angles in a Saccheri quadrilateral are congruent.

The students were assigned to construct a Saccheri quadrilateral by using the Geometry Explorer software. After that, they were asked to measure the two summit angles in the quad and compare their measures. Then, they were asked to make observations for different Saccheri quadrilaterals by dragging their first quadrilaterals. Students made some observations by following the directions. In a short time period, most of the students realized that the two summit angles are equal. Some students tested their conjectures for different Saccheri quadrilaterals, as seen in Figure

2(A) and 2(B). Other students also confirmed this result by making their observations with their own Saccheri quadrilaterals. As students were working individually, the researcher walked around the classroom and assisted students as necessary.



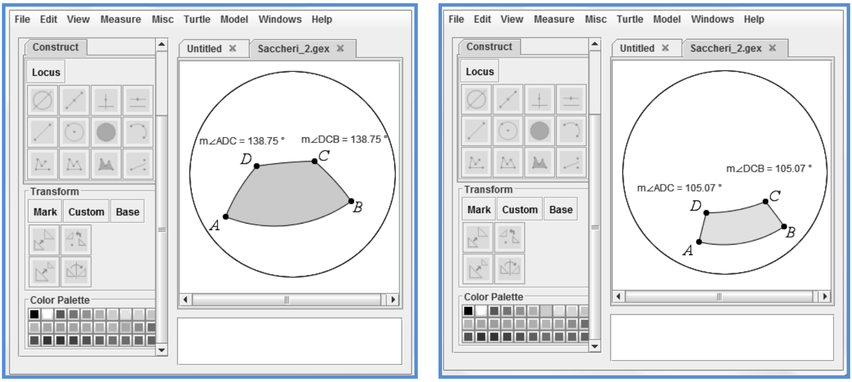
(A) measure of summits angles are 126.41 degrees

(B) measure of summit angles are 113.95 degrees

Figure 2: Saccheri quadrilaterals with angle C and angle D are congruent.

Exploration 2: The summit angles in a Saccheri quadrilateral are obtuse.

The students were asked to construct another Saccheri quadrilateral by using the Geometry Explorer software, measure the two summit angles, angle C and angle D, in the quad, and observe the type of those summit angles whether they are acute, obtuse, or right angles. The students were also asked to make observations for different Saccheri quadrilaterals by dragging their first quadrilaterals. The students made some observations by following the directions. After their observations on the numerical values, most of the students found that the measure of angle C was more than 90 degrees and the measure of angle D was also more than 90 degrees as shown in Figure 3(A) and 3(B). Therefore, they concluded that the two summit angles in a Saccheri quadrilateral are obtuse. Again, as students were working individually, the researcher walked around the classroom and assisted students as necessary.



(A) measure of summits angles are 138.75.41 degrees

(B) measure of summit angles are 105.07 degrees

Figure 3: Saccheri quadrilaterals with angle C and angle D are obtuse angles.

Exploration 3: The line joining the midpoints of the base and summit of a Saccheri quadrilateral is perpendicular to the base and the summit.

The students were asked to construct another Saccheri quadrilateral by using the Geometry Explorer software, create the midpoint of both the base and the summit of a Saccheri quadrilateral, and draw the line joining the midpoints of the base and summit. The students were told that this line is called the altitude of the Saccheri quadrilateral. After that, they were asked to measure the angles between the altitude and the base and the angle between the altitude and the summit. The students were also asked to observe those types of angles. After the observations, most of the students concluded that line joining the midpoints of the base and summit of a

Saccheri quadrilateral made right angles with the base and the summit. It was perpendicular to both as shown in Figure 4.

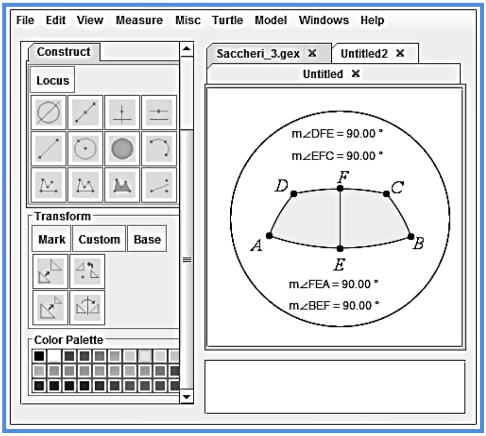
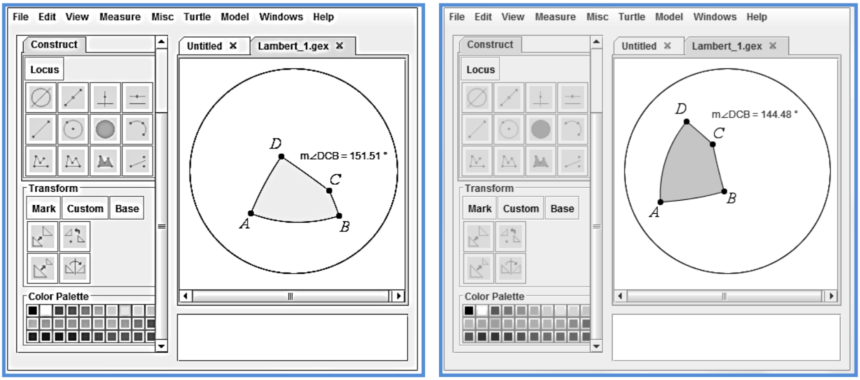


Figure 4: In a Saccheri quadrilateral ABCD, line EF makes right angles with the base AB and the summit CD.

Exploration 4: In a Lambert quadrilateral the fourth angle (the one that is not a right angle) is always obtuse.

The researcher asked the students to draw a Lambert quadrilateral ABCD with right angles at A, B, and D. Then, the researcher asked the students to measure the angle C, the one that is not a right angle and observe the type of that angle. After their observations on the numerical values (Figure 5(A) and 5(B)), most of the students attained the following conjecture:

'The fourth angle in a Lambert quadrilateral is always obtuse.'



(A) measure of angle C is 151.51 degrees (B) measure of angle C is 144.48 degrees

Figure 5: Angle C of a Lambert quadrilateral ABCD is obtuse.

Exploration 5: In a Lambert quadrilateral, each side adjacent to the fourth angle (the one that is not a right angle) has length smaller than the opposite side.

The researcher asked the students to draw another Lambert quadrilateral ABCD with right angles at A, B, and D. Then, the students were asked to measure the sides of their Lambert quadrilaterals. They were asked to compare the lengths of the opposite sides in the Lambert quadrilateral ABCD, and answer the following questions: Are the opposite sides equal? In a pair of opposite sides can you characterize the one which is shorter? After their observations on the numerical values (Figure 6), most of the students attained the following conjecture: 'Each side adjacent to the obtuse angle of a Lambert quadrilateral has length shorter than the opposite side.'

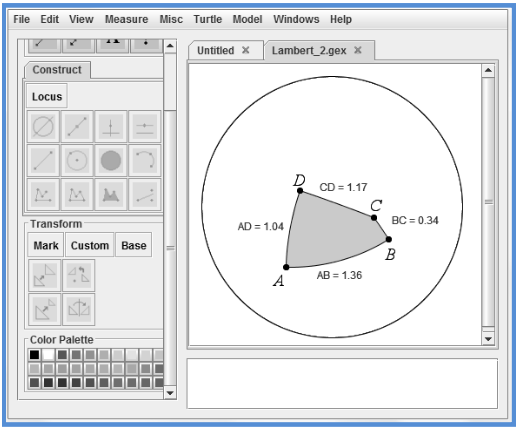


Figure 6: Each side adjacent to the fourth angle of a Lambert quadrilateral has length shorter than the opposite side.

Once the students completed all of the activities, they were asked to compare the properties of quadrilaterals in elliptic geometry with those in Euclidean geometry. The students stated that there were similarities and differences between elliptic geometry and Euclidean geometry. This comparison was shown in Table 1.

Table 1: A Comparison of Euclidean and Elliptic Geometries.

Properties	Euclidean	Elliptic
The summit angles in a Saccheri quadrilateral are	congruent	congruent
The summit angles in a Saccheri quadrilateral are	right angles	obtuse angles
The line joining the midpoints of the base and summit of a Saccheri quadrilateral is	perpendicular to the base and the summit	perpendicular to the base and the summit
In a Lambert quadrilateral the fourth angle is	right angle	obtuse angle
In a Lambert quadrilateral, each side adjacent to the fourth angle has length	equal to the opposite side	shorter than the opposite side

The features of the Geometry explorer that the students often used to identify the properties of quadrilaterals in elliptic geometry were the create panel (point, segment, ray, line), the construct panel (intersection, midpoint, perpendicular, segment on points, circle on points), the measure menu (length, angle, distance). Students often used the drag feature as well.

It was exciting to watch the students conduct the experiment on their own. The researcher had designed the problem-based task such that it enabled the students to be actively involved by giving clear instructions. Meaningful learning became effective whilst students were all actively engaged in problem solving. This became evident when all 26 students obtained 100% in the task to explore Saccheri and Lambert quadrilaterals theorems while using Geometry Explorer. The use of Geometry Explorer eradicated the abstractness the students experienced and provided them with visualization. The “Do it yourself” approach proved to be very effective as compared to demonstrations done by the teacher.

5. Discussion

In this study, 26 students explored important properties of Saccheri and Lambert quadrilaterals of elliptic geometry in the Stereo Graphic Projection model using the Dynamic Geometry program, the Geometry Explorer. Each student individually formed their own examples on computer and compared each other results. Interestingly, they obtained the same results by different examples. Moreover, they had the opportunity to see the different examples of each other since they all formed different ones. The Geometry Explorer allowed students to quickly and easily generate conjectures in elliptic geometry. They made conjectures and verify important properties of the Saccheri and Lambert quadrilateral correctly and rapidly. When the students were asked to compare the properties of these quadrilaterals in elliptic geometry to those in Euclidean geometry, the students could identify the similarities and the differences between the two geometries by using the Geometry Explorer program. These properties were continuously discussed for centuries by Saccheri, Lambert and other mathematicians. While they can easily be explored within Geometry Explorer while dragging and exploring, these properties are certainly out of reach in traditional paper and pencil geometry. As mentioned by Hvidsten (2005), in Geometry Explorer, one can create geometric objects (like points, circles, polygons, areas, and measure aspects of these objects (like length, area, radius, and so on). It is much like doing geometry on paper with a ruler and compass. However, on paper such constructions are static–points placed on the paper can never be moved again. In Geometry Explorer, all constructions are dynamic. One can draw a segment and then grab one of the endpoints and move it around the canvas with the segment moving accordingly. Thus, one can create a construction and test out hypotheses about construction with numerous variations of the original construction. Straesser (2002) also stated that DGS–use widens the range of accessible geometrical constructions and solutions and also widens the range of possible activities, provides an access route to deeper reflection and more refined exploration and heuristics than in paper and pencil geometry. Above all, the students who knew no other geometry other than Euclidean geometry became aware of the existence of other geometries, elliptic geometry. Guven and Karatus (2009) also observed that the DGS turned the geometry classrooms into a laboratory in which students could explore new relations and make conjectures.

6. Conclusion

In the computer–based environment, the students could make conjectures and verify properties of elliptic quadrilaterals correctly and rapidly: the summit angles in a Saccheri quadrilateral are always congruent and obtuse. The line joining the midpoints of the base and summit of a Saccheri quadrilateral is perpendicular to both the base and the summit. In Lambert quadrilateral the fourth angle (the one that is not a right angle) is always obtuse and each side adjacent to this obtuse angle has length smaller than the opposite side. There are similarities and differences between properties of quadrilaterals in Euclidean geometry and elliptic geometry.

7. Implications and Recommendations for Future Research

Implications for Practice

The results of this study indicated that the students could make conjectures and verify properties of elliptic quadrilaterals correctly and rapidly. Thus, exploring mathematical relations and testing conjectures in this dynamic geometry environment make this type of software, Geometry Explorer, a strong learning tool. Therefore, the teacher can use the activity packages with the help of the DGS software as an instructional tool for teaching and learning elliptic geometry in some geometry courses such as the "Introduction to Geometry" course and the "Foundation of Geometry" course.

Even though Geometry Explorer has transformed the classroom environment into a more energetic, dynamically engaging and thought-provoking place, this does not mean that we advocate replacing the use of the real spheres in the classroom with *Geometry Explorer*.

Recommendations for Future Research

According to the results and the limitations of this study, the researcher had some suggestions and recommendations:

1. Some properties of Saccheri and Lambert quadrilaterals in elliptic geometry need to be further investigated: Which is longer, the base or the summit of a Saccheri quadrilateral? Are the diagonals of a Saccheri quadrilateral congruent? Which is longer, the length of the segment joining the midpoints of the summit and base of a Saccheri quadrilateral or each side of the quadrilateral? Is the segment joining the midpoints of the sides of a Saccheri quadrilateral perpendicular to the sides? Is a Saccheri quadrilateral parallelogram? Is a Lambert quadrilateral parallelogram?

2. The participants in this study were 26 students. It is recommended to conduct other studies in the same area with larger samples.

3. Elliptic geometry is certainly not the only type of non-Euclidean geometry in existence. Hence it would be interesting to look into doing similar research for other non-Euclidean geometries such as hyperbolic or spherical geometries.

4. Several studies indicated that students had a positive attitude towards using dynamic geometry programs in mathematics lessons, thus it would also be interesting to examine student attitude towards learning mathematics topics by using the dynamic geometry software, Geometry Explorer.

8. Acknowledgements

I would like to take this opportunity to thank several people who have provided their help and encouragement throughout this study. Appreciation is extended to students who were involved in this study. Without their participation, this study would never have been possible. Finally, acknowledgement is made of the Rajabhat Rajanagarindra University, which provided several supporting. Thanks to all of you.

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