# Confidence Intervals for the Ram Awadh Distribution Parameter: Applications in Engineering and COVID-19 Data

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## Abstract

In applied sciences and medical sciences, accurate estimation of distribution parameter is crucial for data analysis and decision-making. This study proposes various confidence interval (CI) estimation methods for the parameter of the Ram Awadh distribution, a distribution commonly used in reliability and lifetime data analysis. The methods evaluated include likelihood-based, Wald-type, bootstrap-t, and bias-corrected and accelerated (BCa) bootstrap CIs. Through extensive simulation studies, we compare the performance of these methods in terms of empirical coverage probability (ECP) and average length (AL) of the CIs under different sample sizes. Additionally, an explicit formulation of the CI formula for the Wald-type has been derived, simplifying its computation process. The results indicate that both likelihood-based and Wald-type methods consistently approach the nominal confidence level of 0.95 in all scenarios. However, with small sample sizes, the ECP of bootstrap-t and BCa bootstrap CIs tends to decrease. Conversely, as sample sizes increase, the ECPs of all CIs tend to converge

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towards 0.95. Moreover, for all methods, the AL of CIs substantially decreases with increasing sample size. To validate the efficacy of the CIs, they were applied to engineering and COVID-19 data, and the results aligned with those obtained from the simulation studies. In two real-world settings, this thorough evaluation shows that the proposed CIs are reliable and useful for accurately estimating the parameter of the Ram Awadh distribution.

**Keywords:** Interval Estimation, Mixture Distribution, Coverage Probability, Bootstrap Methods, Reliability Data Analysis

# ช่วงความเชื่อมั่นสำหรับพารามิเตอร์ของการแจกแจงราม อวัธ: การประยุกต์กับข้อมูลทางวิศวกรรมและโควิด-19

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**บทคัดย่อ**

ในวิทยาศาสตร์ประยุกต์และวิทยาศาสตร์ทางการแพทย์ การประมาณค่าพารามิเตอร์ของการแจกแจงที่แม่นยำเป็นสิ่งสำคัญสำหรับการวิเคราะห์ข้อมูลและการตัดสินใจ การศึกษานี้เสนอวิธีการประมาณช่วงความเชื่อมั่น (CI) สำหรับพารามิเตอร์ของการแจกแจงราม อวัธ ซึ่งใช้กันทั่วไปในการวิเคราะห์ข้อมูลความน่าเชื่อถือและอายุการใช้งาน วิธีการที่ศึกษา ได้แก่ ช่วงความเชื่อมั่นแบบภาวะน่าจะเป็น ช่วงความเชื่อมั่นแบบวัลด์ ช่วงความเชื่อมั่นแบบบูตสแทร็ป-ที และช่วงความเชื่อมั่นแบบบูตสแทร็ปที่ปรับค่าเอนเอียง ในการเปรียบเทียบประสิทธิภาพของช่วงความเชื่อมั่นพิจารณาจากความน่าจะเป็นคุ้มรวมเชิงประจักษ์ (ECP) และความยาวเฉลี่ย (AL) ของช่วงความเชื่อมั่น ภายใต้ขนาดตัวอย่างที่แตกต่างกัน นอกจากนี้ผู้วิจัยยังสร้างสูตรสำเร็จของช่วงความเชื่อมั่นแบบวัลด์ ซึ่งทำให้การคำนวณง่ายขึ้น ผลลัพธ์แสดงให้เห็นว่าทั้งค่า ECP ของช่วงความเชื่อมั่นแบบภาวะน่าจะเป็น และช่วงความเชื่อมั่นแบบวัลด์เข้าใกล้กับระดับความเชื่อมั่น 0.95 ในทุกสถานการณ์ อย่างไรก็ตาม ค่า ECP ของช่วงความเชื่อมั่นแบบบูตสแทร็ป-ที และช่วงความเชื่อมั่นแบบบูตสแทร็ปที่ปรับค่าเอนเอียงมีแนวโน้มลดลง เมื่อขนาดตัวอย่างเล็ก ในทางกลับกัน เมื่อขนาดตัวอย่างเพิ่มขึ้น ค่า ECP ของทุกช่วงความเชื่อมั่นจะเข้าใกล้กับระดับความเชื่อมั่น 0.95 นอกจากนี้

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ค่า AL ของทุกช่วงความเชื่อมั่นจะลดลงอย่างมีนัยสำคัญ เมื่อขนาดตัวอย่างเพิ่มขึ้น ช่วงความเชื่อมั่นทั้งหมดได้นำไปประยุกต์ใช้กับข้อมูลทางวิศวกรรมและโควิด-19 เพื่อศึกษาผลกระทบของช่วงความเชื่อมั่นและผลลัพธ์สอดคล้องกับการศึกษาเชิงจำลอง จากสถานการณ์จริงสองสถานการณ์แสดงช่วงความเชื่อมั่นที่นำเสนอความน่าเชื่อถือและมีประโยชน์ในการประมาณค่าพารามิเตอร์ของการแจกจแจง ราม อวัธ อย่างแม่นยำ

**คำสำคัญ:** การประมาณค่าแบบช่วง, การแจกแจงแบบผสม, ความน่าจะเป็นคุ้มรวม, วิธีบูตสแทร็ป, การวิเคราะห์ข้อมูลความน่าเชื่อถือ

## Introduction

The study of lifetime distributions is essential across various disciplines, including engineering, insurance, healthcare, and finance. These distributions model the statistical patterns of lifetimes for objects, systems, or events, enabling experts to predict failures, optimize product reliability, determine insurance premiums, plan healthcare interventions, and guide financial decisions. By comprehensively analyzing lifetime distributions, researchers and practitioners can improve decision-making processes, mitigate risks, and efficiently allocate resources, ensuring quality, reliability, and financial stability.

Traditional lifetime distributions, such as the exponential and Lindley distributions, have been widely used in reliability and survival analysis. The exponential distribution assumes a constant hazard rate and possesses the memoryless property, making it mathematically convenient but often unrealistic for many real-world applications. The Lindley distribution, an extension of the exponential distribution, offers greater flexibility with an additional parameter but still may not fully capture the complexities of lifetime behaviors, particularly when the hazard rate varies over time. To address these limitations, researchers have developed more flexible distributions by introducing additional parameters or employing mixture models to better fit empirical data.

Consequently, the development of new flexible probability distributions becomes essential to improve data fitting. Researchers employ several methods to propose new probability distributions or improve the performance of classical distributions. Some of these methods involve adding more parameters to the models, resulting in increased flexibility. Examples include the power Lindley distribution (Ghitany et al., 2013), transmuted Lindley distribution (Merovci, 2013a), transmuted Rayleigh distribution (Merovci, 2013b), Weighted exponential distribution (Dey et al., 2015), generalization of Sujatha distribution (Shanker et al., 2017), power Ishita distribution (Shukla and Shanker, 2018), Darna distribution (Al-Omari et al., 2019), transmuted Aradhana distribution (Gharaibeh, 2020), Loai distribution (Alzoubi et al., 2022a), Sameera distribution (Alzoubi et al., 2022b), among others. However, the approach of adding parameters to the distribution may lead to increased difficulty in parameter estimation.

In recent years, several distributions that better fit the available data have been introduced. Many authors have proposed that mixture distributions be utilized to generate new distributions. For example, Al-Ta'ani and Gharaibeh (2023) suggested the Ola distribution as a three-component mixture of exponential($\boldsymbol{\theta}$), Gamma(4,$\boldsymbol{\theta}$), and Gamma(8,$\boldsymbol{\theta}$), Echebiri and Mbegbu (2022) introduced the Juchez distribution using a three-component mixture of exponential($\boldsymbol{\theta}$), Gamma(2,$\boldsymbol{\theta}$), and Gamma(4,$\boldsymbol{\theta}$).

One notable advancement in this domain is the Ram Awadh distribution, introduced by Shukla (2018). This one-parameter distribution is derived as a mixture of the exponential and gamma distributions, offering enhanced adaptability in modeling real-world lifetime data. Unlike the exponential distribution, which assumes a constant failure rate, the Ram Awadh distribution accommodates a more diverse range of failure patterns by incorporating features of both exponential and gamma distributions. Its flexibility makes it particularly valuable in reliability engineering, medical survival analysis, and risk assessment scenarios where traditional models fail to provide an adequate fit.

In comparison to other mixture-based lifetime distributions, such as the Ola distribution (Al-Ta'ani & Gharaibeh, 2023) and the Juchez distribution (Echebiri & Mbegbu, 2022), the Ram Awadh distribution exhibits several advantages. While the Ola and Juchez distributions are constructed as three-component mixtures of exponential and gamma distributions, the Ram Awadh distribution maintains a simpler structure while achieving comparable or superior fitting capabilities in practical applications. This balance between model simplicity and flexibility makes it an attractive choice for data analysts working with reliability and survival data.

In the review literature, there is no research study for estimating the confidence intervals (CIs) for the parameter of the Ram Awadh distribution. Therefore, this study aims to fill an existing knowledge gap by introducing a method to estimate CIs for the parameter of the Ram Awadh distribution. Four approaches are proposed in this paper: likelihood-based, Wald-type, bootstrap-t, and bias-corrected and accelerated (BCa) bootstrap methods. To assess the performance of these CIs, Monte Carlo simulation studies are conducted, and they are further examined using two real datasets. Furthermore, future research can build upon these findings, potentially extending the CI estimation methods to other distributions and exploring their applications in various fields.

## The Ram Awadh Distribution and Point Parameter Estimation

The Ram Awadh distribution is a flexible alternative for modeling lifetime data. It offers advantages over the traditional exponential and Lindley distributions. Compared to other one-parameter distributions, such as the Ishita distribution (Shanker & Shukla, 2017) and the Pranav distribution (Shukla, 2018b), the Ram Awadh distribution has demonstrated superior goodness-of-fit for certain types of reliability data.

This distribution is obtained by combining the exponential and gamma distributions using appropriate mixing probabilities. The gamma distribution has a shape parameter 6 and a scale parameter $\theta$. Let $X$ be a random variable which follow the Ram Awadh distribution with parameter $\theta$. The probability density function (pdf) of the Ram Awadh distribution can be obtained by using a mixture model with two component mixing probabilities as follows:

$$f(x;\theta) = p_1 g(x;\theta,1) + p_2 g(x;\theta,6) = \frac{\theta^6}{\theta^6+120}\left(\theta e^{-\theta x}\right) + \frac{120}{\theta^6+120}\left(\frac{\theta^6}{120}x^5 e^{-\theta x}\right).$$

Therefore,

$$f(x;\theta) = \frac{\theta^6}{\theta^6+120}(\theta + x^5)e^{-\theta x}, \quad x>0, \theta>0.$$

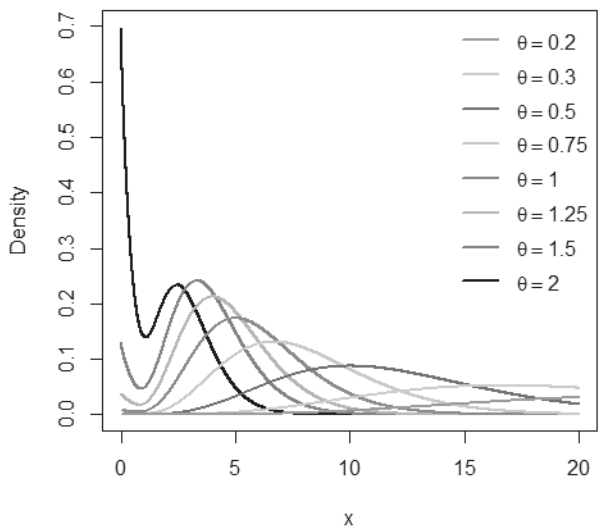



**Figure 1:** The Pdf of the Ram Awadh Distribution for Several Parameter Values

The pdf plot of the Ram Awadh distribution for several parameter values is shown in Figure 1. Additionally, the cumulative distribution function (cdf) for the Ram Awadh distribution is given by

$$F(x;\theta)=1-\left[1+\frac{\theta x(\theta^4x^4+5\theta^3x^3+20\theta^2x^2+60\theta x+120)}{\theta^6+120}\right]e^{-\theta x},\quad x>0,\theta>0.$$

The mean and the $r^{\text{th}}$ non-central moment are as follows:

$$E(X)=\frac{\theta^6+720}{\theta(\theta^6+120)},\text{ and}$$

$$\mu_r'=E(X^r)=\frac{r!\left[\theta^6+(r+1)(r+2)(r+3)(r+4)(r+5)\right]}{\theta^r(\theta^6+120)},\text{ where } r=1,2,3,\ldots$$

The log-likelihood function $\log L(\theta\,|\,x_i)$, is maximized to obtain the point estimator of $\theta$. Therefore, the maximum likelihood (ML) estimator for $\theta$ of the Ram Awadh distribution is derived by the following processes:

$$\begin{aligned}\frac{\partial}{\partial\theta}\log L(\theta\,|\,x_i)&=\frac{\partial}{\partial\theta}\left[6n\log(\theta)-n\log(\theta^6+120)+\sum_{i=1}^{n}\log\left(\theta+x_i^5\right)-\theta\sum_{i=1}^{n}x_i\right]\\&=\frac{6n}{\theta}-\frac{6n\theta^5}{\theta^6+120}+\sum_{i=1}^{n}\left(\frac{1}{\theta+x_i^5}\right)-\sum_{i=1}^{n}x_i.\end{aligned}$$

The subsequent equation is a nonlinear equation obtained through the process of solving the equation $\frac{\partial}{\partial\theta}\log L(x_i;\theta)\overset{\text{set}}{=}0$ for $\theta$,

$$\frac{6n}{\theta}-\frac{6n\theta^5}{\theta^6+120}+\sum_{i=1}^{n}\left(\frac{1}{\theta+x_i^5}\right)-\sum_{i=1}^{n}x_i\overset{\text{set}}{=}0. \tag{1}$$

Since there is no exact mathematical expression available for the ML estimator of the parameter $\theta$, numerical iteration methods are used to solve the related non-linear problem (Nwry et al., 2021). In this study, the maxLik package (Henningsen & Toomet, 2011) was utilized to perform ML estimation using the Newton-Raphson technique in R Studio.

## Confidence Intervals

Confidence interval (CI) estimation is a crucial statistical technique used in parameter estimation. The likelihood-based method, originally proposed by Wilks (1938), provides asymptotically accurate CIs but may require large sample sizes for reliable coverage. The Wald-type CI, an alternative approach, is widely used due to its computational simplicity, although it may be sensitive to small sample sizes and parameter variance (Pawitan, 2001). Bootstrap methods, particularly the bootstrap-t and BCa bootstrap methods, have gained popularity in recent decades as non-parametric alternatives that improve upon the limitations of classical methods (Efron & Tibshirani, 1994). These methods are particularly useful when the sample size is small or when the underlying distribution is unknown.

### Likelihood-Based Confidence Interval

The likelihood-based CI is a statistical tool employed for parameter estimation. This method relies on a likelihood function, which quantifies the probability of observing a given dataset under a specific statistical model and its parameters. This approach's core principle identifies the range where the likelihood function reaches its maximum value, thereby maintaining the desired confidence level. We then maximize the log-likelihood function for the parameter of interest.

Given the observed data $\boldsymbol{x}$ the likelihood function for the Ram Awadh distribution, $L(\theta \mid x)$, is a function of the parameter $\theta$ It measures the probability of observing the given data under different hypothesized $\theta$ values. The ML estimator of $\theta$, $\hat{\theta}$, will be obtained after solving $\frac{\partial}{\partial\theta}\log L(\theta \mid x) \overset{\text{set}}{=} 0$; this estimate is the most likely estimate given the observed data. The likelihood-based CI is then constructed around the ML estimator. This method begins by defining a likelihood ratio $\lambda(\theta)$ as $\lambda(\theta) = L(\theta \mid x) / L(\hat{\theta} \mid x)$. By Wilks' theorem, under regular conditions, the distribution of $-2\log\lambda(\theta)$ follows approximately a chi-square distribution, where the degrees of freedom are equal to the number of parameters being estimated. Thus, the CI for $\theta$ at $(1-\alpha)100\%$ confidence level is given by

$$\left\{\theta\left|-2\log\frac{L(\theta\,|\,x)}{L(\hat{\theta}\,|\,x)}\le\chi^2_{1-\alpha,1}\right.\right\}=$$

$$\left\{\theta\left|-2\log\left[\frac{\theta^{6n}\left(\hat{\theta}^6+120\right)^n}{\hat{\theta}^{6n}\left(\theta^6+120\right)^n}\frac{\prod_{i=1}^{n}(\theta+x_i^5)}{\prod_{i=1}^{n}(\hat{\theta}+x_i^5)}\exp\left(-\theta\sum_{i=1}^{n}x_i+\hat{\theta}\sum_{i=1}^{n}x_i\right)\right]\le\chi^2_{1-\alpha,1}\right.\right\},$$

where $\chi^2_{1-\alpha,1}$ denoted by the critical value obtained from the chi-square distribution with one degree of freedom (Severini, 2000). When considering the Ram Awadh distribution, the likelihood ratio test (LRT) becomes more complicated due to the composite structure of the distribution. The gamma component, which includes shape and scale parameters, adds further complexity to the likelihood function. To address this, computational methods such as numerical optimization are employed to construct CIs and estimate the ML estimator.

We derive the ML estimator of the Ram Awadh distribution using a root-finding approach to optimization based on Brent's method. Given that

$$f(\theta)=\frac{\partial}{\partial\theta}\log L(\theta\,|\,x)=\frac{6n}{\theta}-\frac{6n\theta^5}{\theta^6+120}+\sum_{i=1}^{n}\left(\frac{1}{\theta+x_i^5}\right)-\sum_{i=1}^{n}x_i\overset{\text{set}}{=}0,$$

Using Brent's method, $\theta$ is found so that $f(\theta)=0$. Brent's method benefits from the reliability of bracketing methods, ensuring convergence, and the efficiency of open methods, which typically converge faster. If the product of $f(a)$ and $f(b)$ is less than zero, $f(a)f(b)<0$, the bisection method is applied as an initial step to ensure reliability. Consequently, it alternates between inverse quadratic interpolation and linear or quadratic polynomial interpolation, depending on the behavior of the function and the bracketing interval:

$$\theta_{\text{second}}=\theta_n-f(\theta_n)\frac{\theta_n-\theta_{n-1}}{f(\theta_n)-f(\theta_{n-1})},$$

and secant technique (linear interpolation):

$$\theta_{\text{quad}}=\frac{f(\theta_{n-1})f(\theta_{n-2})}{\left(f(\theta_n)-f(\theta_{n-1})\right)\left(f(\theta_n)-f(\theta_{n-2})\right)}\theta_n+\ldots$$

This method uses an iterative process to improve the root estimate by alternating between several methods, resulting in a more stable and accurate estimate (Kiusalaas, 2013; Bolker, 2023). The plot $-2\log\lambda(\theta)$ versus $\theta$ (solid blue line), $\chi^2_{0.95,1}$ (dashed red line), and 95% likelihood-based CI (solid green line) when a random sample of size 20 sampled from the Ram Awadh distribution with $\theta$=1 is shown in Figure 2.

By Wilks' theorem (Wilks, 1983), the cut-point for constructing a likelihood-based CI often uses an asymptotic distribution, such as the chi-square distribution. The sample size influences the accuracy with which the likelihood-based CI approximates the true parameter values. However, the effectiveness of likelihood-based CI is not strictly dependent on large sample sizes. Even with a small sample size, the likelihood function can generate precise interval estimates, provided it exhibits satisfactory behavior.

**Wald-Type Confidence Interval**

In this subsection, we use the Wald-type CI to estimate the parameter of the Ram Awadh distribution. For the Ram Awadh distribution, the Wald-type CI relies on the parameter's ML estimate, known as $\hat{\theta}$. This CI is constructed using a quadratic approximation of the log-likelihood function, denoted as $L(\theta|x)$, which can be further extended by using a Taylor series around $\hat{\theta}$. The Wald statistic approximates the log-likelihood ratio by expanding to the second-order component around the ML estimate, with the first-order term being zero:

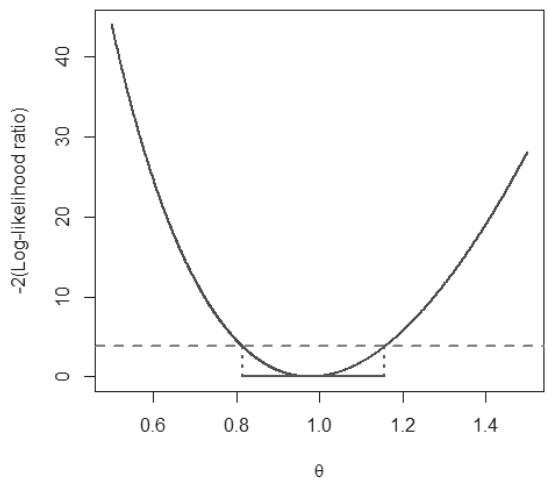



**Figure 2** The plot of $-2\log\lambda(\theta)$ versus $\theta$

$$\log L(\theta \mid x) \approx \log L(\hat{\theta} \mid x) + \left(\theta - \hat{\theta}\right) \frac{\partial}{\partial \theta} \log L(\theta \mid x)\Big|_{\theta=\hat{\theta}} + \frac{1}{2}\left(\theta - \hat{\theta}\right)^2 \frac{\partial^2}{\partial \theta^2} \log L(\theta \mid x)\Big|_{\theta=\hat{\theta}}$$

$$\log \frac{L(\theta \mid x)}{L(\hat{\theta} \mid x)} \approx \frac{1}{2}\left(\theta - \hat{\theta}\right)^2 \frac{\partial^2}{\partial \theta^2} \log L(\theta \mid x)\Big|_{\theta=\hat{\theta}}$$

$$-2 \log \frac{L(\theta \mid x)}{L(\hat{\theta} \mid x)} \approx (\theta - \hat{\theta})^2 I(\hat{\theta}),$$

where $I(\hat{\theta})$ is the observed Fisher information. When the sample size is large enough for the asymptotic properties to hold, one can use the Wald statistic to approximate the LRT statistic, leading to a quadratic approximation of the log-likelihood ratio (Pawitan, 2001). The first and second derivatives of the log-likelihood function for the Ram Awadh distribution are as follows:

$$\frac{\partial}{\partial \theta} \log L(\theta \mid x) = \frac{6n}{\theta} - \frac{6n\theta^5}{\theta^6 + 120} + \sum_{i=1}^{n} \left( \frac{1}{\theta + x_i^5} \right) - \sum_{i=1}^{n} x_i,$$

$$\frac{\partial^2}{\partial \theta^2} \log L(\theta \mid x) = -\frac{6n}{\theta^2} - \frac{30n\theta^4}{\theta^6 + 120} + \frac{36n\theta^{10}}{(\theta^6 + 120)^2} - \sum_{i=1}^{n} \frac{1}{\left(\theta + x_i^5\right)^2}.$$

Therefore, the estimated Fisher information is as follows:

$$I(\hat{\theta}) = E\left[ -\frac{\partial^2}{\partial \theta^2} \log L(\theta \mid x) \mid \hat{\theta} \right] = \frac{6n}{\hat{\theta}^2} + \frac{30n\hat{\theta}^4}{\hat{\theta} + 120} - \frac{36n\hat{\theta}^{10}}{(\hat{\theta} + 120)^2} + n\left( \frac{1}{\hat{\theta}^2 + 2\hat{\theta}\hat{\mu}_5' + \hat{\mu}_{10}'} \right),$$

where $\hat{\mu}_5'$ and $\hat{\mu}_{10}'$ are given by

$$\hat{\mu}_5' = \frac{5!(\hat{\theta}^6 + 30240)}{\hat{\theta}^5(\hat{\theta}^6 + 120)}, \text{ and } \hat{\mu}_{10}' = \frac{10!(\hat{\theta}^6 + 360360)}{\hat{\theta}^{10}(\hat{\theta}^6 + 120)},$$

The Wald-type confidence limits for $\hat{\theta}$ at a confidence level of $(1-\alpha)100\%$ are given by

$$\hat{\theta} \pm z_{1-(\alpha/2)} \sqrt{I^{-1}(\hat{\theta})},$$

where $z_{1-(\alpha/2)}$ denotes the $(1-(\alpha/2))^{\text{th}}$ quantile of the standard normal distribution.

**Bootstrap-T Confidence Interval**

The bootstrap-t CI is an advanced method that calculates the CI for a parameter by considering the variability of the estimate's standard error. This approach improves the accuracy and reliability of the percentile bootstrap method, particularly in situations where the sample size is limited or the estimators do not follow a normal distribution (Panichkitkosolkul, 2024). The procedure for creating a bootstrap-t CI can be explained through the following steps:

1) Initialization: The procedure begins by selecting a random sample $X_1,\ldots,X_n$ and estimating the parameter, $\hat{\theta}$ and its standard error $S.E.(\hat{\theta})$ are obtained.

2) Bootstrap Resampling: Generate $B$ =1000 bootstrap samples, $X_1^*,\ldots,X_n^*$ by random sampling with replacement from the original dataset.

3) Statistical Computation: For each bootstrap sample, compute the bootstrap replicate of the estimator, denoted as $\hat{\theta}^*$ and its associated standard error $S.E.(\hat{\theta})$.

4) Studentization: Construct the bootstrap-t statistic for each replicate as

$$t^*(X,\hat{\theta},\hat{\theta}^*)=\frac{\hat{\theta}^*-\hat{\theta}}{\sqrt{I^{-1}(\hat{\theta}^*)}}.$$

5) Repeating this procedure $B$ =1000 times results in an empirical distribution of the estimator to approximate the distribution of the pivotal quantity.

6) Empirical Distribution: Construct the empirical distribution of the bootstrap-t statistics using the set of $B$ replicates.

7) Quantile Extraction: Find the critical values, $t^*_{(\alpha/2)}$ and $t^*_{(1-(\alpha/2))}$, which correspond to the $(\alpha/2)^{\text{th}}$ and $(1-(\alpha/2))^{\text{th}}$ quantiles of the empirical bootstrap-t distribution,

$$\frac{\#\left(t^*(X,\hat{\theta},\hat{\theta}^*)\le t^*_{\alpha/2}\right)}{B}=\alpha \text{ and } \frac{\#\left(t^*(X,\hat{\theta},\hat{\theta}^*)\le t^*_{1-(\alpha/2)}\right)}{B}=1-(\alpha/2),$$

where $\#(\cdot)$ denotes the number of times the condition is true.

8) Confidence Interval Construction: The bootstrap-t CI is as follows:

$$\hat{\theta}+t^*_{\alpha/2}\sqrt{I^{-1}(\hat{\theta})},\ \hat{\theta}+t^*_{1-(\alpha/2)}\sqrt{I^{-1}(\hat{\theta})}.$$

**Bias-Corrected and Accelerated (BCa) Bootstrap Confidence Interval**

The BCa bootstrap method is a statistical technique used to estimate CIs. This method improves on the basic bootstrap method by incorporating adjustments for both bias and skewness within the distribution of bootstrap estimates. Bias is determined by examining the proportion of bootstrap estimates that fall below the observed estimate, and it is then used to adjust the percentiles of the CI. Additionally, an acceleration parameter is introduced to accommodate the asymmetry or skewness present in the bootstrap distribution. The procedure is described below:

1) Bootstrap Resampling: From the empirical distribution of the original sample, generate $B=1000$ bootstrap samples and compute the bootstrap estimates $\hat{\theta}_b^*$, for $b=1,2,...,1000.$

2) Bias Correction $(z_0)$ Determine the proportion of bootstrap estimates that are less than the original estimate $\hat{\theta}$, denoted $p$. The bias correction factor $z_0$ is the quantile of the standard normal distribution corresponding to $p$.

3) Acceleration $(a)$: Find the acceleration value $a$ that accounts for the estimator's distributional asymmetry. This is often estimated using the jackknife method or other methods that quantify the skewness of the sampling distribution.

4) Adjusted Percentiles: Transform the bias-corrected normal deviates to adjust the percentiles for constructing the CI. The adjusted percentiles are given by

$$p_L^* = \Phi\left( z_0 + \frac{z_0 + z_{\alpha/2}}{1 - a\left(z_0 + z_{\alpha/2}\right)} \right) \text{ and } p_U^* = \Phi\left( z_0 + \frac{z_0 + z_{1-(\alpha/2)}}{1 - a\left(z_0 + z_{1-(\alpha/2)}\right)} \right),$$

where $\Phi$ is the standard normal cumulative distribution function, and $z_{\alpha/2}$ and $z_{1-\alpha/2}$ are the $(\alpha/2)^{\text{th}}$ and $(1-(\alpha/2))^{\text{th}}$ quantiles of the standard normal distribution, respectively.

5) Confidence Interval Construction: The BCa bootstrap CI is constructed using the percentiles $p_L^*$ and $p_U^*$ to extract the corresponding quantiles from the bootstrap distribution of $\hat{\theta}_{ML}^*$. The BCa bootstrap CI is as follows:

$$\left[ \hat{\theta}^*_{(P_L^*)}, \hat{\theta}^*_{(P_U^*)} \right],$$

where $\hat{\theta}^*_{(P_L^*)}$ and $\hat{\theta}^*_{(P_U^*)}$ are $(p_L^*)^{\text{th}}$ and $(p_U^*)^{\text{th}}$ quantiles of the bootstrap estimates $\hat{\theta}_b^*$.

## Simulation Results

### Key Concepts of Monte Carlo Simulations

Monte Carlo simulations (MCS) are a computational technique that utilizes repeated random sampling to estimate numerical results for complex systems involving uncertainty. By assigning probability distributions to input parameters, MCS generates numerous iterations to approximate expected outcomes. The method is widely applied in finance, engineering, physics, artificial intelligence, and healthcare for risk assessment, optimization, and prediction. The key advantages of MCS include handling high-dimensional problems, providing probabilistic insights, and offering flexibility across disciplines. However, limitations such as high computational cost and sensitivity to input distributions must be considered. In statistical inference, MCS plays a crucial role in evaluating confidence intervals and hypothesis testing. In machine learning, it aids Bayesian inference and uncertainty quantification. Its effectiveness improves with increased iterations, ensuring more accurate approximations. As an indispensable tool in modern research, MCS continues to drive advancements in data-driven decision-making (McLeish, 2005).

### Simulation Setting

This paper proposes the 95% two-sided CIs for the parameter of the Ram Awadh distribution. The CIs are determined using four different approaches: likelihood-based, Wald-type, bootstrap-t, and the BCa bootstrap methods. The simulation study evaluates the efficiency of the proposed CIs using R Studio, considering several scenarios. The sample sizes, parameter values, empirical coverage probability (ECP), and the average length (AL) of the CIs are the factors for evaluating the performance of the proposed CIs. Sample sizes ($\boldsymbol{n}$) are set up at 10, 20, 30, 50, 100, 200, and 500, while the distribution's parameter values ($\boldsymbol{\theta}$) are 0.2, 0.3, 0.5, 0.75, 1, 1.25, 1.5, and 2. Monte Carlo simulations are performed with 2,000 replications.

### Empirical Coverage Probability

The results of the simulation study are presented in Table 1 and depicted in Figure 3. The ECPs for likelihood-based and Wald-type CIs approach the nominal level of 0.95 for all cases. This indicates that these CIs exhibit satisfactory performance in terms of ECP. However, the bootstrap-t and BCa bootstrap methods demonstrate lower ECPs, particularly when dealing with small sample sizes.

For all CIs, the sample size has a significant effect on the ECP. For smaller sample sizes, such as $n$= 10 and $n$ = 20, the ECPs are consistently below the nominal level of 0.95 for the bootstrap-t and BCa bootstrap CIs, indicating under-coverage. This under-coverage suggests that these bootstrap methods may not be reliable for small sample sizes, potentially leading to biased or inaccurate inferences. Nevertheless, as the sample size increases, the ECPs of the CIs tend to rise and approach the nominal confidence level. This implies that although these methods are sensitive to sample size, they can still provide adequate coverage in larger samples. The improvement in ECPs with increasing sample size highlights the importance of having a sufficiently large sample to achieve reliable and accurate interval estimates.

Moreover, this convergence towards the nominal level is more rapid for the likelihood-based and Wald-type CIs compared to the bootstrap-t and BCa bootstrap CIs. This rapid convergence indicates that the likelihood-based and Wald-type methods are more robust to smaller sample sizes and can achieve nominal coverage levels more quickly. Therefore, in practical applications, these methods might be preferred when working with limited data.

**Average Length**

The simulation results for the AL of the CIs are shown in Table 1 and depicted graphically in Figure 4. For all methods, the AL of CIs decreases significantly as the sample size increases. This trend is consistent and expected, as larger sample sizes typically provide more information about the parameter of the distribution, thereby reducing the estimate's uncertainty. This reduction in AL with increasing sample size indicates improved precision of the interval estimates. When the sample size is small, the CIs are wider, reflecting greater uncertainty in the parameter estimates. Conversely, as the sample size increases, the CIs narrow, signifying increased precision in the estimates and increased confidence in the parameter estimation.

Additionally, the rate of decrease in AL varies among different CI methods. Methods that are more efficient in utilizing the information from the sample tend to exhibit a faster reduction in AL. For instance, likelihood-based and Wald-type CIs might show a steeper decline in AL compared to bootstrap methods, reflecting their greater efficiency in certain scenarios.

**Table 1:** Empirical Coverage Probability and Average Length of the 95% CIs for the Parameter of the Ram Awadh Distribution

| $\theta$ | $n$ | Empirical Coverage Probability | | | | Average Length | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | Likelihood | Wald | Boot-t | BCa | Likelihood | Wald | Boot-t | BCa |
| 0.20 | 10 | 0.9475 | 0.9510 | 0.8965 | 0.8980 | 0.1023 | 0.1021 | 0.0924 | 0.0964 |
| | 20 | 0.9530 | 0.9490 | 0.9230 | 0.9265 | 0.0722 | 0.0721 | 0.0686 | 0.0699 |
| | 30 | 0.9400 | 0.9445 | 0.9205 | 0.9225 | 0.0589 | 0.0588 | 0.0567 | 0.0575 |
| | 50 | 0.9490 | 0.9510 | 0.9395 | 0.9465 | 0.0453 | 0.0453 | 0.0444 | 0.0449 |
| | 100 | 0.9505 | 0.9515 | 0.9460 | 0.9445 | 0.0321 | 0.0321 | 0.0316 | 0.0319 |
| | 200 | 0.9530 | 0.9515 | 0.9505 | 0.9545 | 0.0227 | 0.0227 | 0.0224 | 0.0225 |
| | 500 | 0.9515 | 0.9500 | 0.9450 | 0.9450 | 0.0143 | 0.0143 | 0.0142 | 0.0143 |
| 0.30 | 10 | 0.9485 | 0.9515 | 0.9010 | 0.8865 | 0.1547 | 0.1545 | 0.1401 | 0.1461 |
| | 20 | 0.9495 | 0.9500 | 0.9225 | 0.9255 | 0.1084 | 0.1083 | 0.1029 | 0.1047 |
| | 30 | 0.9500 | 0.9455 | 0.9285 | 0.9270 | 0.0880 | 0.0880 | 0.0845 | 0.0858 |
| | 50 | 0.9440 | 0.9435 | 0.9360 | 0.9380 | 0.0681 | 0.0681 | 0.0662 | 0.0668 |
| | 100 | 0.9560 | 0.9545 | 0.9490 | 0.9495 | 0.0480 | 0.0480 | 0.0475 | 0.0479 |
| | 200 | 0.9500 | 0.9505 | 0.9465 | 0.9470 | 0.0340 | 0.0340 | 0.0338 | 0.0340 |
| | 500 | 0.9465 | 0.9475 | 0.9450 | 0.9450 | 0.0215 | 0.0215 | 0.0213 | 0.0215 |
| 0.50 | 10 | 0.9490 | 0.9480 | 0.8970 | 0.8895 | 0.2581 | 0.2577 | 0.2328 | 0.2427 |
| | 20 | 0.9475 | 0.9445 | 0.9240 | 0.9275 | 0.1797 | 0.1796 | 0.1715 | 0.1748 |
| | 30 | 0.9510 | 0.9530 | 0.9350 | 0.9330 | 0.1471 | 0.1470 | 0.1427 | 0.1444 |
| | 50 | 0.9435 | 0.9405 | 0.9340 | 0.9310 | 0.1134 | 0.1134 | 0.1107 | 0.1118 |
| | 100 | 0.9540 | 0.9535 | 0.9470 | 0.9475 | 0.0802 | 0.0802 | 0.0793 | 0.0800 |
| | 200 | 0.9480 | 0.9485 | 0.9445 | 0.9430 | 0.0567 | 0.0567 | 0.0561 | 0.0566 |
| | 500 | 0.9430 | 0.9430 | 0.9425 | 0.9405 | 0.0358 | 0.0358 | 0.0356 | 0.0358 |
| 0.75 | 10 | 0.9500 | 0.9530 | 0.8985 | 0.8940 | 0.3828 | 0.3827 | 0.3440 | 0.3558 |
| | 20 | 0.9555 | 0.9550 | 0.9335 | 0.9265 | 0.2693 | 0.2693 | 0.2565 | 0.2602 |
| | 30 | 0.9535 | 0.9520 | 0.9375 | 0.9380 | 0.2196 | 0.2196 | 0.2127 | 0.2152 |
| | 50 | 0.9440 | 0.9470 | 0.9350 | 0.9360 | 0.1696 | 0.1696 | 0.1651 | 0.1667 |
| | 100 | 0.9425 | 0.9460 | 0.9390 | 0.9420 | 0.1198 | 0.1198 | 0.1180 | 0.1192 |
| | 200 | 0.9535 | 0.9530 | 0.9510 | 0.9535 | 0.0847 | 0.0847 | 0.0839 | 0.0847 |
| | 500 | 0.9425 | 0.9420 | 0.9400 | 0.9425 | 0.0534 | 0.0534 | 0.0530 | 0.0534 |

**Table 1:** Empirical Coverage Probability and Average Length of the 95% CIs for the Parameter of the Ram Awadh Distribution (Continue)

| $\theta$ | $n$ | Empirical Coverage Probability | | | | Average Length | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | Likelihood | Wald | Boot-t | BCa | Likelihood | Wald | Boot-t | BCa |
| 1.00 | 10 | 0.9480 | 0.9485 | 0.8920 | 0.8890 | 0.4974 | 0.4993 | 0.4471 | 0.4520 |
| | 20 | 0.9450 | 0.9465 | 0.9200 | 0.9170 | 0.3507 | 0.3513 | 0.3322 | 0.3341 |
| | 30 | 0.9500 | 0.9515 | 0.9365 | 0.9355 | 0.2863 | 0.2867 | 0.2760 | 0.2776 |
| | 50 | 0.9490 | 0.9480 | 0.9355 | 0.9390 | 0.2219 | 0.2222 | 0.2157 | 0.2172 |
| | 100 | 0.9550 | 0.9550 | 0.9465 | 0.9480 | 0.1570 | 0.1571 | 0.1557 | 0.1568 |
| | 200 | 0.9560 | 0.9575 | 0.9500 | 0.9530 | 0.1108 | 0.1109 | 0.1096 | 0.1104 |
| | 500 | 0.9515 | 0.9515 | 0.9515 | 0.9510 | 0.0701 | 0.0701 | 0.0696 | 0.0702 |
| 1.25 | 10 | 0.9470 | 0.9430 | 0.8845 | 0.8885 | 0.5814 | 0.5861 | 0.5291 | 0.5270 |
| | 20 | 0.9530 | 0.9500 | 0.9245 | 0.9220 | 0.4138 | 0.4160 | 0.3960 | 0.3941 |
| | 30 | 0.9570 | 0.9550 | 0.9415 | 0.9375 | 0.3385 | 0.3399 | 0.3278 | 0.3274 |
| | 50 | 0.9525 | 0.9540 | 0.9430 | 0.9435 | 0.2626 | 0.2634 | 0.2579 | 0.2577 |
| | 100 | 0.9470 | 0.9475 | 0.9455 | 0.9475 | 0.1860 | 0.1865 | 0.1840 | 0.1848 |
| | 200 | 0.9435 | 0.9435 | 0.9400 | 0.9385 | 0.1316 | 0.1319 | 0.1308 | 0.1317 |
| | 500 | 0.9575 | 0.9565 | 0.9570 | 0.9560 | 0.0833 | 0.0835 | 0.0827 | 0.0835 |
| 1.50 | 10 | 0.9565 | 0.9545 | 0.8915 | 0.8860 | 0.6353 | 0.6403 | 0.5756 | 0.5815 |
| | 20 | 0.9525 | 0.9510 | 0.9315 | 0.9260 | 0.4509 | 0.4538 | 0.4313 | 0.4314 |
| | 30 | 0.9535 | 0.9540 | 0.9335 | 0.9285 | 0.3690 | 0.3711 | 0.3569 | 0.3572 |
| | 50 | 0.9480 | 0.9495 | 0.9370 | 0.9370 | 0.2864 | 0.2878 | 0.2796 | 0.2806 |
| | 100 | 0.9530 | 0.9550 | 0.9415 | 0.9455 | 0.2026 | 0.2036 | 0.2002 | 0.2015 |
| | 200 | 0.9440 | 0.9450 | 0.9445 | 0.9470 | 0.1433 | 0.1440 | 0.1424 | 0.1431 |
| | 500 | 0.9440 | 0.9440 | 0.9390 | 0.9415 | 0.0907 | 0.0911 | 0.0906 | 0.0912 |
| 2.00 | 10 | 0.9565 | 0.9645 | 0.9010 | 0.9340 | 0.7318 | 0.7318 | 0.6533 | 0.7332 |
| | 20 | 0.9425 | 0.9495 | 0.9245 | 0.9355 | 0.5076 | 0.5108 | 0.4785 | 0.5051 |
| | 30 | 0.9635 | 0.9685 | 0.9485 | 0.9590 | 0.4119 | 0.4154 | 0.3964 | 0.4115 |
| | 50 | 0.9420 | 0.9470 | 0.9345 | 0.9420 | 0.3174 | 0.3206 | 0.3090 | 0.3170 |
| | 100 | 0.9560 | 0.9590 | 0.9515 | 0.9520 | 0.2237 | 0.2262 | 0.2213 | 0.2245 |
| | 200 | 0.9545 | 0.9565 | 0.9505 | 0.9510 | 0.1577 | 0.1596 | 0.1566 | 0.1585 |
| | 500 | 0.9505 | 0.9555 | 0.9475 | 0.9475 | 0.0997 | 0.1010 | 0.0990 | 0.1000 |

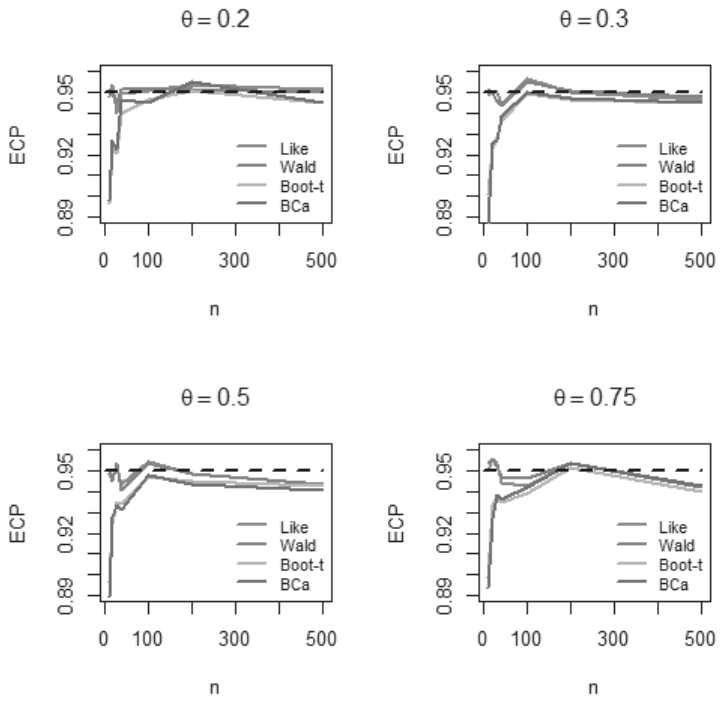



**Figure 3:** Plots of the ECPs of the CIs for $\theta$ of the Ram Awadh Distribution

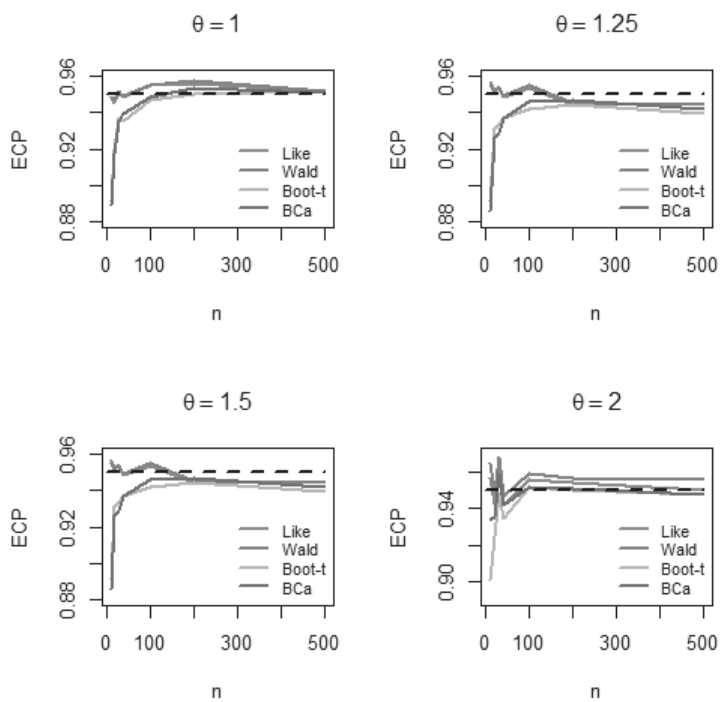



**Figure 3:** Plots of the ECPs of the CIs for $\theta$ of the Ram Awadh Distribution (Continued)

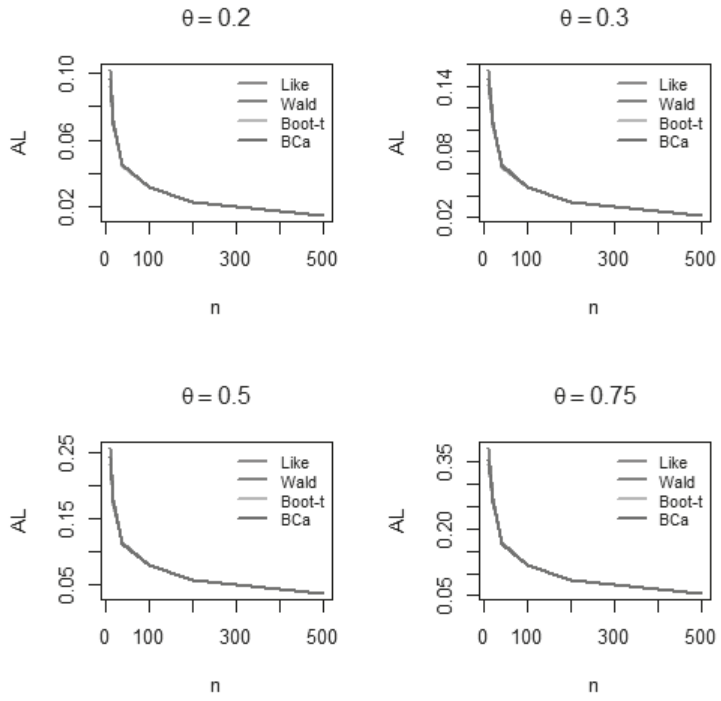



**Figure 4:** Plots of the ALs of the CIs for $\theta$ of the Ram Awadh Distribution

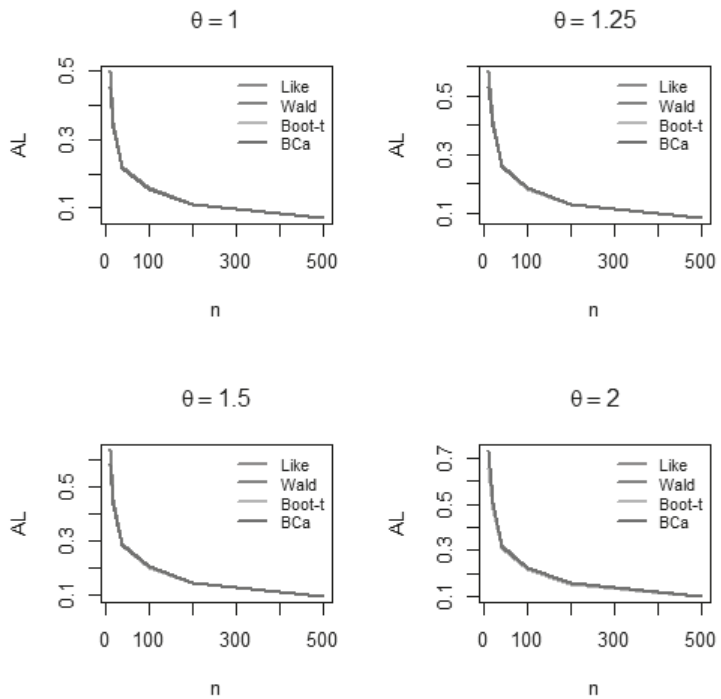



**Figure 4:** Plots of the ALs of the CIs for $\theta$ of the Ram Awadh Distribution (Continued)

## Applications to Real Data

Four CIs for the parameter of the Ram Awadh distribution, as proposed in the previous section, are applied to a real-world situation. The adequacy of the Ram Awadh distribution's performance is being compared to that of the following alternative distributions:

1) The Pratibha distribution (Shanker, 2023)

$$f(x;\theta)=\frac{\theta^3}{\theta^3+\theta+2}\left(\theta+x+x^2\right)e^{-\theta x},\quad x>0,\theta>0.$$

2) The Chris-Jerry distribution (Onyekwere and Obulezi, 2022)

$$f(x;\theta)=\frac{\theta^2}{\theta+2}\left(1+\theta x^2\right)e^{-\theta x},\quad x>0,\theta>0.$$

3) The Juchez distribution (Echebiri and Mbegbu, 2022)

$$f(x;\theta)=\frac{\theta^4}{\theta^3+\theta^2+6}(1+x+x^3)e^{-\theta x},\quad x>0,\theta>0.$$

4) The Kpenadidum distribution (Nwikpe and Cleopas, 2022)

$$f(x;\theta)=\frac{\theta^4}{2(\theta^4+\theta+6)}(2x^3+x^2+2\theta)e^{-\theta x},\ x>0,\theta>0.$$

5) The Zeghdoudi distribution (Messaadia and Zeghdoudi, 2018)

$$f(x;\theta)=\frac{\theta^3}{\theta+2}x(1+x)e^{-\theta x},\quad x>0,\theta>0.$$

6) The Pranav distribution (Shukla, 2018b)

$$f(x;\theta)=\frac{\theta^4}{\theta^4+6}(\theta+x^3)e^{-\theta x},\ x>0,\theta>0.$$

7) The Rani distribution (Shanker, 2017a)

$$f(x;\theta)=\frac{\theta^5}{\theta^5+24}(\theta+x^4)e^{-\theta x},\ x>0,\theta>0.$$

8) The Rama distribution (Shanker, 2017b)

$$f(x;\theta)=\frac{\theta^4}{\theta^3+6}(1+x^3)e^{-\theta x},\ x>0,\theta>0.$$

9) The Suja distribution (Shanker, 2017c)

$$f(x;\theta)=\frac{\theta^5}{\theta^4+24}(1+x^4)e^{-\theta x},\ \ x>0,\theta>0.$$

10) The Ishita distribution (Shanker and Shukla, 2017)

$$f(x;\theta)=\frac{\theta^3}{\theta^3+2}(\theta+x^2)e^{-\theta x},\ \ x>0,\theta>0.$$

11) The Devya distribution (Shanker, 2016a)

$$f(x;\theta)=\frac{\theta^5}{(\theta^4+\theta^3+2\theta^2+6\theta+24)}(1+x+x^2+x^3+x^4)e^{-\theta x},\ \ x>0,\theta>0.$$

12) The Garima distribution (Shanker, 2016b)

$$f(x;\theta)=\frac{\theta}{\theta+2}(1+\theta+\theta x)e^{-\theta x},\ \ x>0,\theta>0.$$

13) The Aradhana distribution (Shanker, 2016c)

$$f(x;\theta)=\frac{\theta^3}{\theta^2+2\theta+2}(1+x)e^{-\theta x},\quad x>0,\theta>0.$$

14) The Akash distribution (Shanker, 2015)

$$f(x;\theta)=\frac{\theta^3}{\theta^2+2}(1+x^2)e^{-\theta x},\ \ x>0,\theta>0.$$

15) The Lindley distribution (Lindley, 1958)

$$f(x;\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x},\ \ x>0,\theta>0.$$

16) The exponential distribution

$$f(x;\theta)=\theta e^{-\theta x},\ \ x>0,\theta>0.$$

**Engineering Data**

Fuller et al. (1994) reported the strength data of the aircraft window glass. The dataset comprises the following data: 18.83, 20.80, 21.657, 23.03, 23.23, 24.05, 24.321, 25.50, 25.52, 25.80, 26.69, 26.77, 26.78, 27.05, 27.67, 29.90, 31.11, 33.20, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381. Table 2 presents descriptive statistics for this dataset, which provide a summary of the dataset and highlight key aspects such as central tendency, variability, and distribution shape.

**Table 2:** Descriptive Statistics for the Strength data of the Aircraft Window Glass

| Sample Sizes | Minimum | Q1 | Median | Mean | Q3 | Maximum | St.Dev |
|---|---|---|---|---|---|---|---|
| 31 | 18.8300 | 25.5100 | 29.9000 | 30.8114 | 35.8300 | 45.3810 | 7.2534 |

The ML method was utilized to estimate all parameters of the distributions under consideration. The study conducted a comprehensive evaluation using various criteria, including the log-likelihood (log L), Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC), to compare the goodness of fit of different distributions. The combined use of log L, AIC, and BIC provides a robust comparative analysis, allowing for the identification of the most appropriate model for the given dataset. Table 3 presents the parameter estimates along with their corresponding standard errors (SEs) and measures of goodness of fit for the dataset under investigation.

**Table 3:** Analysis of Model Fit for Several Mixture Distributions Applied to the Strength Data of the Aircraft Window Glass

| Distributions | Estimates (SE) | Log L | AIC | BIC |
|---|---|---|---|---|
| Ram Awadh | 0.1947 ($2.039\times10^{-4}$) | -111.5392 | 225.0783 | 226.5123 |
| Pratibha | 0.0955 ($9.892\times10^{-5}$) | -121.2989 | 244.5977 | 246.0317 |
| Chris-Jerry | 0.1296 ($1.352\times10^{-4}$) | -116.4447 | 234.8894 | 236.3234 |
| Juchez | 0.1291 ($1.345\times10^{-4}$) | -116.5236 | 235.0472 | 236.4812 |
| Kpenadidum | 0.0959 ($9.892\times10^{-5}$) | -120.6531 | 243.3062 | 244.7402 |
| Zeghdoudi | 0.1298 ($1.359\times10^{-4}$) | -116.3876 | 234.7752 | 236.2092 |
| Pranav | 0.1623 ($1.699\times10^{-4}$) | -113.6252 | 229.2503 | 230.6843 |
| Rani | 0.1298 ($1.358\times10^{-4}$) | -116.3962 | 234.7924 | 236.2263 |
| Rama | 0.1623 ($1.699\times10^{-4}$) | -113.6259 | 229.2517 | 230.6857 |
| Suja | 0.0973 ($1.018\times10^{-4}$) | -120.2435 | 242.4871 | 243.9210 |
| Ishita | 0.1609 ($1.668\times10^{-4}$) | -113.8427 | 229.6854 | 231.1194 |
| Devya | 0.0516 ($6.220\times10^{-5}$) | -133.6390 | 269.2780 | 270.7120 |
| Garima | 0.1283 ($1.326\times10^{-4}$) | -116.7044 | 235.4087 | 236.8427 |
| Aradhana | 0.0971 ($1.010\times10^{-4}$) | -120.3409 | 242.6818 | 244.1157 |
| Akash | 0.0647 ($6.727\times10^{-5}$) | -127.0170 | 256.0339 | 257.4679 |
| Lindley | 0.0325 ($3.398\times10^{-5}$) | -137.2644 | 276.5289 | 277.9629 |
| Exponential | 0.1947 ($2.039\times10^{-4}$) | -111.5392 | 225.0783 | 226.5123 |

*Note.* The boldface denotes the distribution with the lowest AIC and BIC values.

The log-likelihood, AIC, and BIC values in Table 3 indicate that the Ram Awadh distribution offers a suitable fit compared to other distributions under consideration. The maximum likelihood (ML) estimator for this dataset is 0.1947, suggesting a robust fit of the distribution to the observed data. In Table 4, the 95% two-sided CIs for the parameter of the Ram Awadh distribution provide additional insights into the precision of the parameter estimate. The likelihood-based method generates a CI spanning from 0.1681 to 0.2241, with an interval length of 0.0560, reflecting a level of uncertainty in the parameter estimate. Similarly, the Wald-type method produces a CI of 0.1667 to 0.2227, demonstrating consistency with the likelihood-based method in both range and uncertainty, with the same interval length of 0.0560. Conversely, both the bootstrap-t method and the BCa bootstrap method yield notably narrow CIs, suggesting decreased uncertainty in the parameter estimate. This difference emphasizes the importance of method selection and highlights potential variations in precision when estimating the parameter of the Ram Awadh distribution.

**Table 4:** Comparison of 95% Two-Sided CIs for the Strength Data of the Aircraft Window Glass

| CI Methods | CIs | Lengths |
|---|---|---|
| Likelihood-based | (0.1681, 0.2241) | 0.0560 |
| Wald-type | (0.1667, 0.2227) | 0.0560 |
| Bootstrap-t | (0.1787, 0.2122) | 0.0335 |
| BCa bootstrap | (0.1780, 0.2099) | 0.0319 |

**COVID-19 Data**

The number of new deaths caused by COVID-19 in Jordan from January 31, 2021 to February 28, 2021 were reported by World Health Organization (WHO) and Al-Ta'ani and Gharaibeh (2023). The observations are: 12, 10, 8, 10, 10, 8, 7, 10, 6, 10, 16, 10, 12, 11, 11, 18, 18, 12, 9, 16, 15, 11, 16, 19, 22, 16, 2,3 25, 26. Detailed descriptive statistics for this dataset are presented in Table 5.

**Table 5:** Descriptive Statistics for the Number of New Deaths Caused by COVID-19 in Jordan

| Sample Sizes | Minimum | Q1 | Median | Mean | Q3 | Maximum | St.Dev |
|---|---|---|---|---|---|---|---|
| 29 | 6.00 | 10.00 | 12.00 | 13.69 | 16.00 | 26.00 | 5.4321 |

All parameters of the distributions were estimated by the ML method. We consider the log-likelihood (log L), AIC, and BIC to compare several distributions. Table 6 provides estimates of the parameters, their SEs, and measures of goodness of fit for this dataset.

**Table 6:** Analysis of Model Fit for Several Mixture Distributions Applied to the Number of New Deaths Caused by COVID-19 in Jordan

| Distributions | Estimates (SE) | Log L | AIC | BIC |
|---|---|---|---|---|
| Ram Awadh | 0.4383 ($1.104\times10^{-3}$) | -87.4442 | 176.8883 | 178.2556 |
| Pratibha | 0.2114 ($5.112\times10^{-4}$) | -92.3166 | 186.6332 | 188.0005 |
| Chris-Jerry | 0.2096 ($5.186\times10^{-4}$) | -93.4929 | 188.9858 | 190.3531 |
| Juchez | 0.2893 ($7.128\times10^{-4}$) | -89.5929 | 181.1858 | 182.5531 |
| Kpenadidum | 0.2885 ($7.158\times10^{-4}$) | -89.5308 | 181.0616 | 182.4289 |
| Zeghdoudi | 0.2121 ($5.189\times10^{-4}$) | -92.2371 | 186.4741 | 187.8414 |
| Pranav | 0.2919 ($7.317\times10^{-4}$) | -89.3547 | 180.7093 | 182.0766 |
| Rani | 0.3651 ($9.185\times10^{-4}$) | -88.0756 | 178.1511 | 179.5184 |
| Rama | 0.2913 ($7.270\times10^{-4}$) | -89.4217 | 180.8434 | 182.2107 |
| Suja | 0.3650 ($9.173\times10^{-4}$) | -88.0873 | 178.1747 | 179.5420 |
| Ishita | 0.2181 ($5.414\times10^{-4}$) | -91.7003 | 185.4006 | 186.7679 |
| Devya | 0.3574 ($8.763\times10^{-4}$) | -88.3679 | 178.7357 | 180.1030 |
| Garima | 0.1135 ($3.235\times10^{-4}$) | -102.1145 | 206.2290 | 207.5963 |
| Aradhana | 0.2045 ($4.815\times10^{-4}$) | -93.0343 | 188.0687 | 189.4360 |
| Akash | 0.2158 ($5.278\times10^{-4}$) | -92.0459 | 186.0917 | 187.4590 |
| Lindley | 0.1440 ($3.507\times10^{-4}$) | -97.3532 | 196.7064 | 198.0737 |
| Exponential | 0.0730 ($1.840\times10^{-4}$) | -104.8826 | 211.7651 | 213.1324 |

*Note:* The Boldface Denotes the Distribution with the Lowest AIC and BIC Values.

Table 7 compares 95% two-sided CIs and their lengths for parameter estimation using a variety of methods. The likelihood-based method estimates the CI to be between 0.3763 and 0.5066, with an interval length of 0.1303. The Wald-type method produces a CI that ranges from 0.3732 to 0.5034, with a length of 0.1302, closely aligning with the results from the likelihood-based method. On the other hand, the BCa bootstrap method offers a narrower CI, spanning from 0.3776 to 0.5056, with a reduced interval length of 0.1280. Similarly, the bootstrap-t method provides CI ranging from 0.3800 to 0.5028, with the shortest interval length at 0.1228.

**Table 7:** Comparison of 95% Two-Sided CIs for the Number of New Deaths Caused by COVID-19 in Jordan

| CI Methods | CIs | Lengths |
|---|---|---|
| Likelihood-based | (0.3763, 0.5066) | 0.1303 |
| Wald-type | (0.3732, 0.5034) | 0.1302 |
| Bootstrap-t | (0.3776, 0.5056) | 0.1280 |
| BCa bootstrap | (0.3800, 0.5028) | 0.1228 |

From two real datasets, these results highlight the precision and reliability trade-offs between various CI estimate techniques. The likelihood-based and Wald-type methods, which are both parametric, give similar CI ranges and lengths, which suggests that they give about the same level of accuracy for this set of data. However, the slightly narrower CI produced by the BCa bootstrap method indicates a potential for increased accuracy without significant loss in terms of interval length. The fact that this non-parametric method can give a shorter CI shows how well it can handle the data's underlying distributional properties.

The bootstrap-t method stands out by generating the shortest CI, which suggests a high level of precision. This method's efficiency in providing the tightest interval indicates its effectiveness in situations where a more exact estimation is crucial. Nonetheless, when interpreting these results, it is important to consider the context of the data and the assumptions underlying each method. The differences in interval lengths can have significant implications in practical applications, particularly in fields where parameter estimation plays a critical role in decision-making and policy formulation.

## Conclusion and Discussion

This paper presents and evaluates four distinct methods for constructing confidence intervals (CIs) for the parameters of the Ram Awadh distribution using likelihood-based, Wald-type, bootstrap-t, bias-corrected and accelerated (BCa) bootstrap methods. The explicit formula for the Wald-type CI was derived and proposed in this paper. Simulation studies evaluate all CIs by investigating their empirical coverage probability (ECP) and the average length (AL) of the intervals. The simulation study results indicate that

likelihood-based and Wald-type CIs achieve the nominal confidence level of 0.95 across all cases, showing satisfactory performance. In contrast, bootstrap-t and BCa bootstrap methods exhibit lower ECPs, particularly with small sample sizes, leading to under-coverage and potential inaccuracies. As sample sizes increase, the ECPs for all CIs improve, approaching the nominal confidence level, emphasizing the need for larger samples for reliable estimates. Likelihood-based and Wald-type CIs converge more quickly to the nominal confidence level, suggesting they are more robust and preferable for small sample sizes in practical applications.

In summary, while bootstrap-t and BCa bootstrap methods offer flexibility and potential advantages in certain contexts, their performance is highly dependent on sample size. For smaller samples, alternative methods like likelihood-based and Wald-type CIs may be more appropriate to ensure accurate and reliable coverage. As sample size increases, the bootstrap methods become more competitive, suggesting their utility in larger datasets where their advantages can be fully realized. This analysis underscores the importance of considering sample size when selecting a method for constructing CIs, to balance between robustness and flexibility. Furthermore, the simulation results clearly demonstrate that increasing sample size leads to narrower CIs across all methods, enhancing the precision of parameter estimates. This underscores the importance of having a sufficiently large sample size in statistical analysis to achieve more reliable and precise estimates.

The proposed CI estimation methods for the Ram Awadh distribution have significant real-world applications, particularly in healthcare data monitoring and engineering system reliability analysis. In healthcare, they assist in epidemiological modeling, such as estimating COVID-19 fatality trends, guiding clinical trials, and improving survival analysis for disease treatments. In engineering, they enhance predictive maintenance, optimize quality control, and assess structural reliability, as demonstrated in aircraft window glass strength analysis. These methods help estimate failure probabilities and ensure safety in high-risk industries. Their adaptability extends to financial risk assessment, environmental modeling, and supply chain optimization, highlighting their broad impact. Future research can refine these methods for industry-specific needs, ensuring accurate uncertainty estimation for better decision-making.

The bootstrap-t and BCa bootstrap methods rely on the fundamental assumption that resampled data faithfully represents the characteristics of the underlying population. However, bootstrap methods face several challenges when applied to small sample sizes. Increased variability in resampling can produce unstable intervals, while heavy reliance on the initial sample increases the risk of overfitting. Resampling instability prevents accurate representation of the full distribution, especially for skewed or heavy-tailed data. Mean estimation bias may occur, and BCa corrections might be insufficient, leading to asymmetric CIs. Small samples also struggle to approximate higher-order moments, affecting interval accuracy. Additionally, bootstrap methods may fail to capture extreme values, reducing their robustness in limited data scenarios.

Additionally, the computational demands of bootstrap methods may pose significant constraints, particularly in scenarios with limited computational resources. Fortunately, several packages in the R programming language offer solutions to simplify the computation of bootstrap CIs, exemplified by the 'boot' package (Canty and Ripley, 2024) and the 'bootstrap' package (Kostyshak, 2024). These packages provide efficient algorithms to expedite the calculation process, thereby enhancing the accessibility and usability of bootstrap methods in statistical analysis.

To improve bootstrap performance in small samples, future research should explore adaptive bootstrap techniques, such as double bootstrap or shrinkage-based resampling, to enhance coverage probabilities. Hybrid methods that combine parametric and non-parametric approaches, like Bayesian bootstrapping, could improve robustness. Extensive simulations across various sample sizes and distribution shapes can help establish optimal sample size thresholds. Alternative resampling strategies, such as smoothed bootstrap or residual-based resampling, may address small-sample biases. Investigating higher-order corrections and bias-reduction methods can further refine interval accuracy. Additionally, developing computationally efficient bootstrap algorithms can reduce processing time while maintaining reliability. These advancements will ensure more accurate and applicable bootstrap-based CIs across different scenarios.

Furthermore, future research could investigate the CIs for the parameters of other weighted mixed distributions, such as the weighted Pranav (Shukla and Shanker, 2020), weighted Nwikpe (Mohiuddin et al., 2022), and weighted Odoma (Manoj and Elangovan, 2020), among others.

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