A Note on Calculus: Bisection Point Theorem

Wanwisa Puangmalai, Paichayon Sirisathianwatthana, Nirutt Pipattanajinda*
Program of Mathematics, Faculty of Sciences and Technology, Kamphaeng Phet Rajabhat University
*Corresponding author E-mail: nirutt.p@gmail.com

Abstract

In this note, we defined the bisection point, the point β on a closed interval $[a, b]$ with a continuous and nonnegative function of real numbers such that the value of the definite integral between the $[a, \beta]$ and $[\beta, b]$ is equal. Furthermore, we present several examples to demonstrate how to find the bisection point on a nonnegative polynomial function.
Keywords: Calculus, bisection point, area problem.

Introduction and Preliminaries

In 1854, Bernhard Riemann introduced the method known as the Riemann Integral, which revolutionized calculus by providing a rigorous framework for finding the area under a curve. This concept is the key concept of this note.

For the importance of this problem, let's first consider a right triangle ABC with $|AB| = |BC| = 4$ in Figure 1, the area of ABC is $\frac{|AB||BC|}{2} = 8$. If β is a point on the segment AB such that the perpendicular line halves the area of ABC, then the right triangle of base $A\beta$ is 4. Thus $4 = \frac{|A\beta|^2}{2}$ or $|A\beta| = 2\sqrt{2}$. However, if AC is a section of the curve $\frac{x^2}{4}$, see Figure 2, it is not possible to use geometric methods to determine to value of β. Therefore, we use calculus to solve the problem. We recall the fundamental theorem of calculus.

Theorem 1. (The Fundamental Theorem of Calculus, part II) If f is a real-valued continuous function on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

From Theorem 1, if f is also a nonnegative function, $F(b) - F(a)$ becomes the area enclosed by the curve f and the x-axis from a to b. Therefore, we use the integration to get some facts of the value β in $[a, b]$ such that the area between f and the x-axis from a to β and β to b are equal.

Received: Feb. 2, 2024 **Revised:** Jun. 28, 2024 **Accepted:** Jun. 28, 2024 **Published:** Jun. 30, 2024

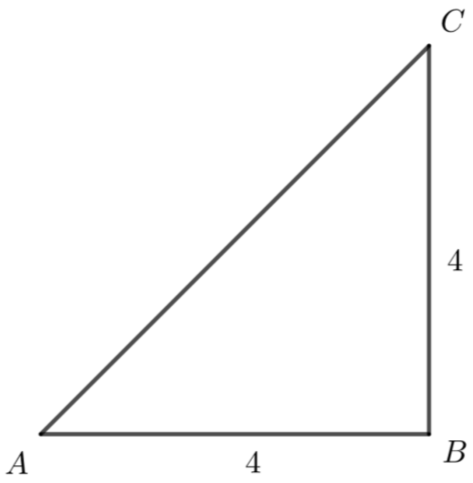


Fig. 1: Right triangle ABC

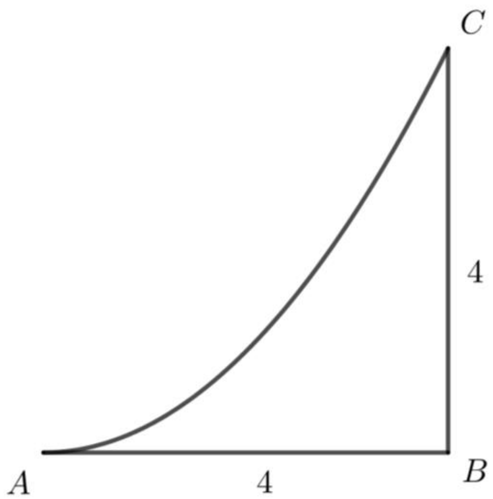


Fig. 2: The modifies AC to curve

Main Results

First, we show theorem of bisection point for confirming the existence of the bisection point of half area between f and the x-axis.

Theorem 2. (The Bisection Point Theorem) Let $f:[a,b]\to\mathbb{R}$ be a continuous and nonnegative function and F is an antiderivative of f. If F is injective, then there exists uniquely $\beta\in[a,b]$ such that,

$$F(\beta)=\frac{F(a)+F(b)}{2}.$$

Proof. Let $\int_a^b f(x)dx=A\in\mathbb{R}^+$. Then there exists $\beta\in[a,b]$ such that

$$\int_a^\beta f(x)dx=\int_\beta^b f(x)dx=\frac{A}{2}.$$

If there exists $\beta_1,\beta_2\in[a,b]$ such that

$$\int_a^{\beta_1} f(x)dx=\int_{\beta_1}^b f(x)dx=\frac{A}{2}, \text{ and } \int_a^{\beta_2} f(x)dx=\int_{\beta_2}^b f(x)dx=\frac{A}{2}.$$

Thus, $\int_a^{\beta_1} f(x)dx=\int_a^{\beta_2} f(x)dx.$ So, $F(\beta_1)=F(\beta_2)$. Since, F is injective, $\beta_1=\beta_2$.

Then there exists uniquely $\beta\in[a,b]$ such that

$$\int_a^\beta f(x)dx=\int_\beta^b f(x)dx.$$

From theorem 1, $F(\beta)-F(a)=F(b)-F(\beta)$. □

From Theorem 2, if the antiderivative F of f isn't injective, then the value $\beta\in[a,b]$ such that $F(\beta)=\frac{F(a)+F(b)}{2}$ is no need to have a single value. For example, $f(x)=0$, we have $F(\beta)=\frac{F(a)+F(b)}{2}$ for any $\beta\in[a,b]$. So we get the Corollary 1 as the following.

Corollary 1. Let $f:[a,b] \to \mathbb{R}$ be a continuous and nonnegative function and F is an antiderivative of f. Then there exists $\beta \in [a,b]$ such that $F(\beta) = \frac{F(a)+F(b)}{2}$.

From Theorem 2, the definition of a bisection point is the following.

Definition 1. Let $f:[a,b] \to \mathbb{R}$ be a continuous and nonnegative function and F is an antiderivative. The point $\beta \in [a,b]$ is said to be a **bisection point** of $[a,b]$, if $F(\beta) = \frac{F(a)+F(b)}{2}$.

In the case of the simple polynomial $f(x) = mx^{n-1}$ where $n \in \mathbb{N}$, f is a continuous and nonnegative function on $[0,\infty)$, forming an area akin to a triangle with a curved section. We have discovered some interesting results regarding bisection points that depend on the degree n, but not on the coefficient m.

Theorem 3. Let $m \in \mathbb{R}$ and $n \in \mathbb{N}$. If $f(x) = mx^{n-1}$ and β is a bisection point of $[0,a]$, then $\beta = \frac{a}{\sqrt[n]{2}}$.

Proof. Let F be an antiderivative of f with a bisection point $\beta \in [0,a]$, then $\frac{m\beta^n}{n} = \frac{ma^n}{2n}$. Thus $\beta = \frac{a}{\sqrt[n]{2}}$. □

For example, if $f(x) = mx$ is the right triangle in $[0,1]$, then the bisection point is always $\beta = \frac{1}{\sqrt{2}}$ for arbitrary m, see Figure 3. Further, we show the solution of Figures 1-2, the bisection points of x and $\frac{x^2}{4}$ are $\frac{4}{\sqrt{2}}$ and $\frac{4}{\sqrt[3]{2}}$, respectively, see Figure 4. Furthermore, if $f(x) = mx^{n-1}$ at $n \to \infty$ and β is a bisection point of $[0,a]$, then $\beta = \lim_{n\to\infty} \frac{a}{\sqrt[n]{2}} = a$.

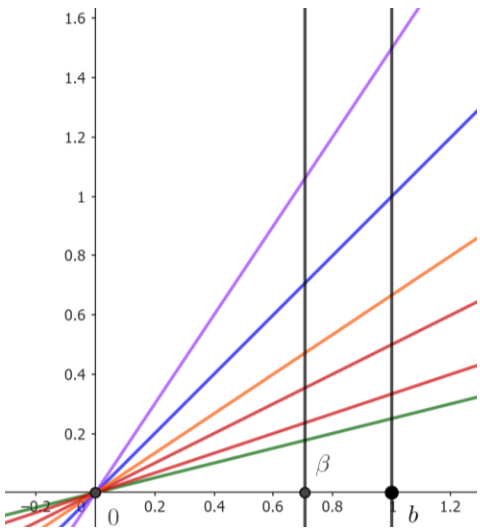
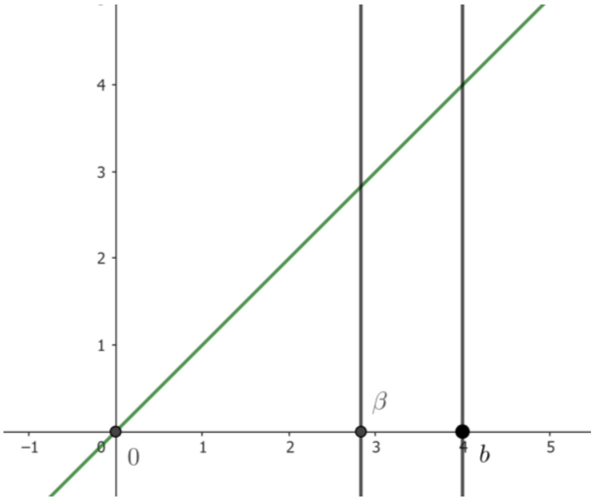


Fig. 3: The unique bisection points of right triangle $f(x) = mx$ on $[0,1]$, where $m \in \{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1, \frac{3}{2}\}$



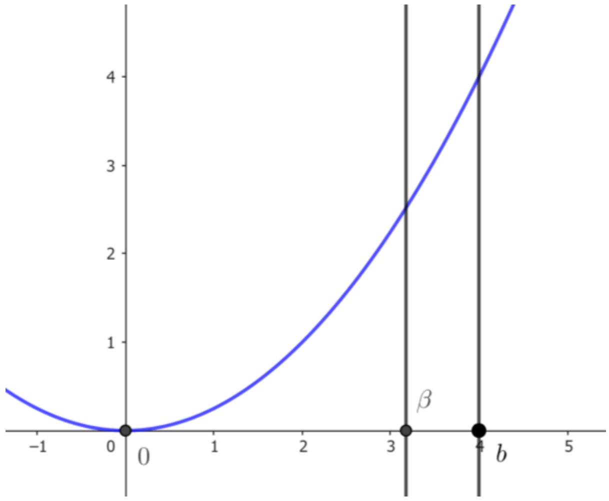


Fig. 4: The bisection points of $\boldsymbol{x}$ and $\frac{\boldsymbol{x}^2}{4}$

Next, we consider on the linear polynomial $f(x) = Ax + B$ on $[0, a]$. If $\beta \in [0, a]$ is the bisection point, by Definition 1, we have $\frac{A\beta^2}{2} + B\beta = \frac{Aa^2}{4} + \frac{Ba}{2}$. Thus $\frac{A\beta^2}{2} + B\beta - (\frac{Aa^2}{4} + \frac{Ba}{2}) = 0$, and implies that $\beta = \left(-B \pm \sqrt{B^2 + \frac{A^2a^2}{2} + ABa}\right)/A$. So, inferring that the bisection point is the complex pattern, if f is binomial with terms more than 1. We will use the SciLab program (Ver. 6.1.0) to present examples for estimating bisection points of polynomial functions in addition to mx^{n-1} for any interval $[a, b]$, following as Table 5.

Table 1: Examples for estimating bisection points of some polynomial functions.

$\boldsymbol{f(x)}$	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{\beta}$ (estimate)	$\boldsymbol{F(\beta) - F(a)}$	$\boldsymbol{F(b) - F(\beta)}$
$\boldsymbol{(x-1)^3}$	1.5	5	4.363791	31.992189	31.992185
$\boldsymbol{-x^3 + 3x^2 + 3x + 5}$	-1	3	1.670268	19.999999	20.000000
$\boldsymbol{x^4 - 3x^3 + 5x^2 - 7x + 9}$	0	2	0.854617	5.866665	5.866667
$\boldsymbol{3x^5 - 7x^4 - 24x^3 + 6x^2 + 20x + 12}$	-1.8	-0.2	-1.264942	41.182208	41.182207
$\boldsymbol{-4x^6 + x^5 - x^4 - 3x^3 + 3x^2 - x + 10}$	-1	1	-0.107619	10.228571	10.228570
$\boldsymbol{5x^7 - 2x^6 - 6x^5 + x^4 - 3x^3 + 3x^2 + 4x + 2}$	-1	1	0.321644	2.914283	2.914288

From Table 5, we can use Theorem 2 for any polynomial function on small interval $[a, b]$ and β is the estimated bisection point, such that $F(\beta) - F(a) \approx F(b) - F(\beta)$, but for a large distance interval $[a, b]$, the bisection point is difficult to find. The estimated value to be close to the bisection point is necessary.

Conclusion

In this note, we defined and proved the existence of bisection points for continuous and nonnegative functions. We presented formulas for determining the bisection points of the polynomial $f(x) = mx^{n-1}$. Finally, numerical examples were provided to illustrate finding bisection points on nonnegative polynomial functions. However, for other types of functions where bisection points are not explicitly estimated, an algorithm for estimation can identify bisection points within any interval.

Acknowledgments

The authors gratefully acknowledge the referee for their helpful comments and suggestions.

References

Boyer, Carl B., and Uta C. Merzbach. A History of Mathematics. John Wiley & Sons, 2011.

Strang G. and Herman, E. "Calculus Volume 1." Rice University. https://assets.openstax.org/oscms-prodcms/media/documents/Calculus_Volume_1_-_WEB_68M1Z5W.pdf

Weisstein, Eric W. "Calculus." From MathWorld--A Wolfram Web Resource. https://mathworld.wolfram.com/Calculus.html

Weisstein, Eric W. "Second Fundamental Theorem of Calculus." From MathWorld--A Wolfram Web Resource. https://mathworld.wolfram.com/SecondFundamentalTheoremofCalculus.html